

112 學年度第 2 學期

統計學 (二)

Anderson's Statistics for Business & Economics (14/E)

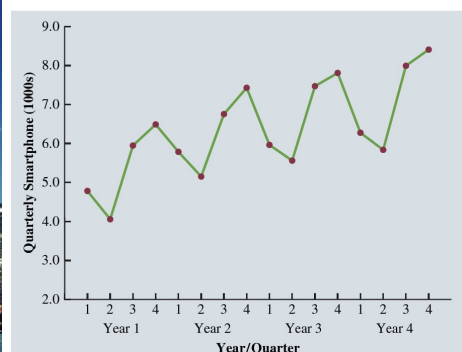
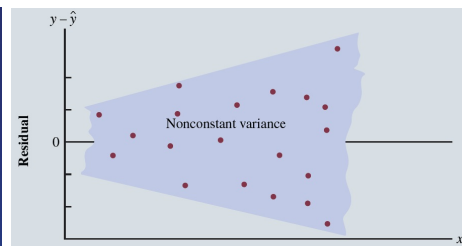
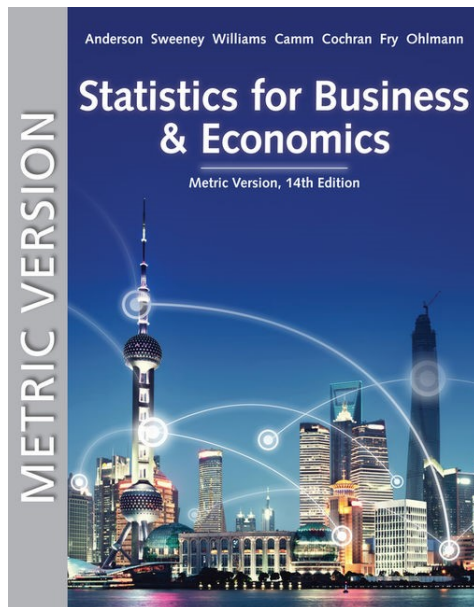
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附錄：111 學年第 2 學期小考題、期中考題、期末考題。

叮嚀

- A. 平常就要唸書，做習題。
- B. 考過的題目，要主動訂正。
- C. 上課以「互相尊重」為最高原則並盡到「告知老師」的義務。
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統計學 (二)

Anderson's Statistics for Business & Economics (14/E)

Chapter 9: Hypothesis Tests

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Overview

1. Statistical inference: how hypothesis testing can be used to determine whether a statement about the value of a population parameter should or should not be rejected.
2. The null hypothesis (H_0): making a tentative assumption about a population parameter.
3. The alternative hypothesis (H_a): the opposite of what is stated in H_0 .
4. The hypothesis testing procedure uses data from a sample to test the two competing statements indicated by H_0 and H_a .
5. This chapter shows how hypothesis tests can be conducted about a population mean and a population proportion.

9.1 Developing Null and Alternative Hypotheses

1. It is not always obvious how the null and alternative hypotheses should be formulated.
2. All hypothesis testing applications involve collecting a sample and using the sample results to provide evidence for drawing a conclusion.
3. In some situations it is easier to identify H_a first and then develop H_0 .
4. In other situations it is easier to identify H_0 first and then develop H_a .

The Alternative Hypothesis as a Research Hypothesis

1. Many applications of hypothesis testing involve an attempt to gather evidence in support of a research hypothesis. In these situations, it is often best to begin with the alternative hypothesis and make it the conclusion that the researcher hopes to support.
2. **Example** Consider a particular automobile that currently attains a fuel efficiency of 24 miles per gallon in city driving.
 - (a) *Goal:* A product research group has developed a new fuel injection system (燃料噴射系統) designed to increase the miles-per-gallon rating. The group will run controlled tests with the new fuel injection system looking for statistical support for the conclusion that the new fuel injection system provides more miles per gallon than the current system.
 - (b) Several new fuel injection units will be manufactured, installed in test automobiles, and subjected to research-controlled driving conditions.
 - (c) The sample mean miles per gallon for these automobiles will be computed and used in a hypothesis test to determine if it can be concluded that the new system provides more than 24 miles per gallon.
 - (d) In terms of the population mean miles per gallon μ , the research hypothesis $\mu > 24$ becomes the alternative hypothesis.

- (e) Since the current system provides an average or mean of 24 miles per gallon, we will make the tentative assumption that the new system is not any better than the current system and choose $\mu \leq 24$ as the null hypothesis.

$$\underline{H_0 : \mu \leq 24, \quad H_a : \mu > 24}$$

- (f) If the sample results lead to the conclusion to reject H_0 , the inference can be made that $H_a : \mu > 24$ is true.
- (g) The researchers have the statistical support to state that the new fuel injection system increases the mean number of miles per gallon.
- (h) If the sample results lead to the conclusion that H_0 cannot be rejected, the researchers cannot conclude that the new fuel injection system is better than the current system. Production of automobiles with the new fuel injection system on the basis of better gas mileage cannot be justified. Perhaps more research and further testing can be conducted.
3. Before adopting something new (e.g., products, methods, systems), it is desirable to conduct research to determine if there is statistical support for the conclusion that the new approach is indeed better. In such cases, the research hypothesis is stated as the alternative hypothesis.
- (a) **Example** A new teaching method is developed that is believed to be better than the current method.
- H_0 : the new method is no better than the old method.
 - H_a : the new method is better.
- (b) **Example** A new sales force bonus plan is developed in an attempt to increase sales.
- H_0 : the new bonus plan does not increase sales.
 - H_a : the new bonus plan increases sales.
- (c) **Example** A new drug is developed with the goal of lowering blood pressure more than an existing drug.
- H_0 : the new drug does not provide lower blood pressure than the existing drug.

- ii. H_a : the new drug lowers blood pressure more than the existing drug.
4. In each case, rejection of the null hypothesis H_0 provides statistical support for the research hypothesis.

The Null Hypothesis as an Assumption to Be Challenged

- The situations below that it is helpful to develop the null hypothesis first.
 - Consider applications of hypothesis testing where we begin with a belief or an assumption that a statement about the value of a population parameter is true.
 - We will then use a hypothesis test to challenge the assumption and determine if there is statistical evidence to conclude that the assumption is incorrect.
- The null hypothesis H_0 expresses the belief or assumption about the value of the population parameter. The alternative hypothesis H_a is that the belief or assumption is incorrect.
- Example** Consider the situation of a manufacturer of soft drink products.
 - The label on a soft drink bottle states that it contains 67.6 fluid ounces. We consider the label correct provided the population mean filling weight for the bottles is at least 67.6 fluid ounces.
 - We would begin with the assumption that the label is correct and state the null hypothesis as $\mu \geq 67.6$.
 - The challenge to this assumption would imply that the label is incorrect and the bottles are being under-filled. This challenge would be stated as the alternative hypothesis $\mu < 67.6$.

$$\underline{H_0 : \mu \geq 67.6 \quad H_a : \mu < 67.6}$$
 - A government agency with the responsibility for validating manufacturing labels could select a sample of soft drinks bottles, compute the sample mean filling weight, and use the sample results to test the preceding hypotheses.

- (e) If the sample results lead to the conclusion to reject H_0 , the inference that $H_a : \mu < 67.6$ is true can be made. With this statistical support, the agency is justified in concluding that the label is incorrect and underfilling of the bottles is occurring.
- (f) If the sample results indicate H_0 cannot be rejected, the assumption that the manufacturer's labeling is correct cannot be rejected. With this conclusion, no action would be taken.
4. **Example** Consider the soft drink bottle filling example from the manufacturer's point of view.
- (a) The bottle-filling operation has been designed to fill soft drink bottles with 67.6 fluid ounces as stated on the label.
- The company does not want to underfill the containers because that could result in an underfilling complaint from customers or, perhaps, a government agency.
 - However, the company does not want to overfill containers either because putting more soft drink than necessary into the containers would be an unnecessary cost.
- (b) The company's goal would be to adjust the bottle-filling operation so that the population mean filling weight per bottle is 67.6 fluid ounces as specified on the label.
- (c) In a hypothesis testing application, we would begin with the assumption that the production process is operating correctly and state the null hypothesis as $\mu = 67.6$ fluid ounces.
- (d) The alternative hypothesis that challenges this assumption is that $\mu \neq 67.6$, which indicates either overfilling or underfilling is occurring.

$$\underline{H_0 : \mu = 67.6 \quad H_a : \mu \neq 67.6.}$$

- (e) Suppose that the soft drink manufacturer uses a quality control procedure to periodically select a sample of bottles from the filling operation and computes the sample mean filling weight per bottle.

- i. If the sample results lead to the conclusion to reject H_0 , the inference is made that $H_a : \mu \neq 67.6$ is true. We conclude that the bottles are not being filled properly and the production process should be adjusted to restore the population mean to 67.6 fluid ounces per bottle.
 - ii. If the sample results indicate H_0 cannot be rejected, the assumption that the manufacturer's bottle filling operation is functioning properly cannot be rejected. In this case, no further action would be taken and the production operation would continue to run.
5. The two preceding forms of the soft drink manufacturing hypothesis test show that the null and alternative hypotheses may vary depending upon the point of view of the researcher or decision maker.
 6. To correctly formulate hypotheses it is important to understand the context of the situation and structure the hypotheses to provide the information the researcher or decision maker wants.

Summary Of Forms for Null and Alternative Hypotheses

1. Depending on the situation, hypothesis tests about a population parameter (the population mean and the population proportion) may take one of three forms:

$H_0 : \mu \geq \mu_0$	$H_0 : \mu \leq \mu_0$	$H_0 : \mu = \mu_0$
$H_a : \mu < \mu_0$	$H_a : \mu > \mu_0$	$H_a : \mu \neq \mu_0$
2. The first two forms are called one-tailed tests. The third form is called a two-tailed test.
3. The equality part of the expression (either \geq , \leq , or $=$) always appears in the null hypothesis.
4. In selecting the proper form of H_0 and H_a , keep in mind that the alternative hypothesis is often what the test is attempting to establish. Hence, asking whether the user is looking for evidence to support $\mu < \mu_0$, $\mu > \mu_0$, or $\mu \neq \mu_0$ will help determine H_a .

9.2 Type I and Type II Errors

1. Ideally the hypothesis testing procedure should lead to the acceptance of H_0 when H_0 is true and the rejection of H_0 when H_a is true.
2. (Table 9.1) The correct conclusions are not always possible.

TABLE 9.1 Errors and Correct Conclusions in Hypothesis Testing

		Population Condition	
		H_0 True	H_a True
Conclusion	Accept H_0	Correct Conclusion	Type II Error
	Reject H_0	Type I Error	Correct Conclusion

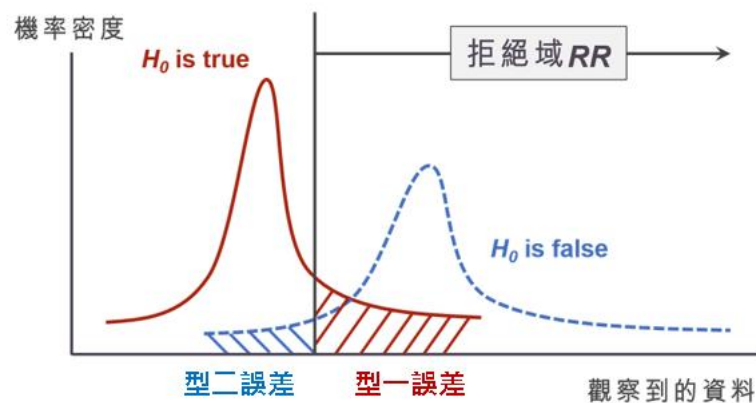
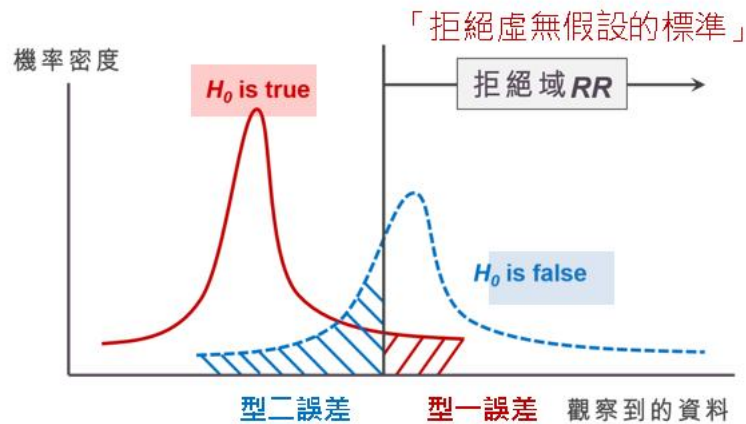
- (a) We reject H_0 if H_0 is true, we make a Type I error.
 - (b) If H_a is true, the conclusion is correct when we reject H_0 .
 - (c) If H_0 is true, the conclusion is correct when we accept H_0 .
 - (d) If H_0 is false (H_a is true), we make a Type II error when we accept H_0 .
3. **Example** An automobile product research group developed a new fuel injection system designed to increase the miles-per-gallon rating of a particular automobile.
 - (a) With the current model obtaining an average of 24 miles per gallon, the hypothesis test was formulated as follows.

$$\underline{H_0 : \mu \leq 24 \text{ against } H_a : \mu > 24}$$

- (b) The alternative hypothesis, $H_a : \mu > 24$, indicates that the researchers are looking for sample evidence to support the conclusion that the population mean miles per gallon with the new fuel injection system is greater than 24.
- (c) *Type I error*: rejecting H_0 when it is true corresponds to the researchers claiming that the new system improves the miles-per-gallon rating ($\mu > 24$) when in fact the new system is not any better than the current system.

- (d) *Type II error*: accepting H_0 when it is false corresponds to the researchers concluding that the new system is not any better than the current system ($\mu \leq 24$) when in fact the new system improves miles-per-gallon performance.
4. **Level of Significance**: The level of significance is the probability of making a Type I error when the null hypothesis is true as an equality.
- (a) **Example** For the miles-per-gallon rating hypothesis test, the null hypothesis is $H_0 : \mu \leq 24$. Suppose the null hypothesis is true as an equality; that is, $\mu = 24$. The level of significance is the probability of rejecting $H_0 : \mu \leq 24$ when $\mu = 24$.
- (b) The Greek symbol α (alpha) is used to denote the level of significance, and common choices for α are 0.05 and 0.01.
- (c) In practice, the person responsible for the hypothesis test specifies the level of significance. By selecting α , that person is controlling the probability of making a Type I error.
- (d) If the cost of making a Type I error is high (not too high), small (larger) values of α are preferred.
5. **The significance tests**: Applications of hypothesis testing that only control for the Type I error are called significance tests.
6. Although most applications of hypothesis testing control for the probability of making a Type I error, they do not always control for the probability of making a Type II error.
- (a) Hence, if we decide to accept H_0 , we cannot determine how confident we can be with that decision. Because of the uncertainty associated with making a Type II error when conducting significance tests, statisticians usually recommend that we use the statement "do not reject H_0 " instead of "accept H_0 ".
- (b) Using the statement "do not reject H_0 " carries the recommendation to withhold both judgment and action. In effect, by not directly accepting H_0 , the statistician avoids the risk of making a Type II error.

- (c) Whenever the probability of making a Type II error has not been determined and controlled, we will not make the statement "accept H_0 ." In such cases, only two conclusions are possible: do not reject H_0 or reject H_0 .
- (d) Although controlling for a Type II error in hypothesis testing is not common, it can be done. In Sections 9.7 and 9.8 we will illustrate procedures for determining and controlling the probability of making a Type II error. If proper controls have been established for this error, action based on the "accept H_0 " conclusion can be appropriate.



9.3 Population Mean: σ Known

One-tailed Test

1. One-tailed tests about a population mean take one of the following two forms:

Lower Tail Test	Upper Tail Test
$H_0 : \mu \geq \mu_0$	$H_0 : \mu \leq \mu_0$
$H_a : \mu < \mu_0$	$H_a : \mu > \mu_0$

2. Example The Federal Trade Commission (FTC) periodically conducts statistical studies designed to test the claims that manufacturers make about their products. For example, the label on a large can of Hilltop Coffee states that the can contains 3 pounds of coffee. The FTC knows that Hilltop's production process cannot place exactly 3 pounds of coffee in each can, even if the mean filling weight for the population of all cans filled is 3 pounds per can ($\mu_0 = 3$). However, as long as the population mean filling weight is at least 3 pounds per can, the rights of consumers will be protected. Thus, the FTC interprets the label information on a large can of coffee as a claim by Hilltop that the population mean filling weight is at least 3 pounds per can. We will show how the FTC can check Hilltop's claim by conducting a lower tail hypothesis test.

- (a) *Develop the null and alternative hypotheses for the test.* If the population mean filling weight is at least 3 pounds per can, Hilltop's claim is correct:

$$H_0 : \underline{\mu \geq 3} \quad H_a : \underline{\mu < 3}$$

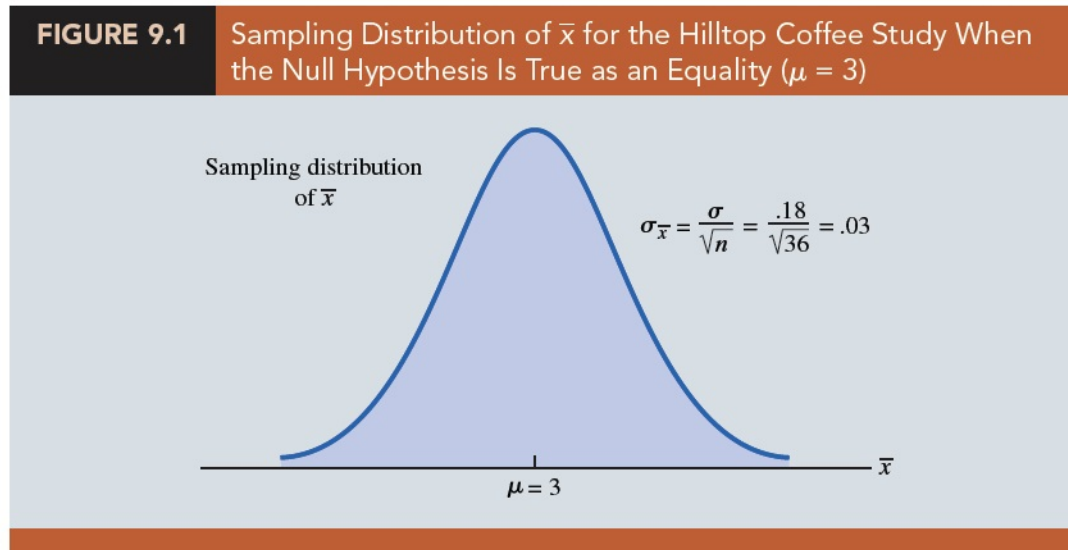
- i. If the sample data indicate that H_0 cannot be rejected, no action should be taken against Hilltop.
- ii. If the sample data indicate H_0 can be rejected, $H_a : \mu < 3$, is true. A conclusion of underfilling and a charge of a label violation against Hilltop would be justified.
- iii. Suppose a sample of $n = 36$ cans of coffee is selected and the sample mean \bar{x} is computed as an estimate of the population mean μ . If the value of the sample mean \bar{x} is less than 3 pounds, the sample results will cast doubt on the null hypothesis.

- (b) *Specifying the level of significance, α :*
- i. (Recall) The level of significance is the probability of making a Type I error by rejecting H_0 when H_0 is true as an equality.
 - ii. If the cost of making a Type I error is high (not high), a small (larger) value should be chosen for the level of significance.
 - iii. In the Hilltop Coffee study, the director of the FTC's testing program made the following statement: "If the company is meeting its weight specifications at $\mu = 3$, I do not want to take action against them. But, I am willing to risk a 1% chance of making such an error." From the director's statement, we set the level of significance for the hypothesis test at $\alpha = 0.01$.
 - iv. Thus, we must design the hypothesis test so that the probability of making a Type I error when $\mu = 3$ is 0.01.
3. By developing the null and alternative hypotheses and specifying the level of significance for the test, we carry out the first two steps required in conducting every hypothesis test. We are now ready to perform the third step of hypothesis testing: collect the sample data and compute the value of what is called a test statistic.

Test statistic

1. Example For the Hilltop Coffee study, previous FTC tests show that the population standard deviation can be assumed known with a value of $\sigma = 0.18$. These tests also show that the population of filling weights can be assumed to have a normal distribution.
2. The sampling distribution of \bar{x} is normally distributed with a known value of $\sigma = 0.18$ and a sample size of $n = 36$.
3. (Figure 9.1) the sampling distribution of \bar{x} when the null hypothesis is true as an equality ($\mu = \mu_0 = 3$). The standard error of \bar{x} is given by $\sigma_{\bar{x}} = \underline{\sigma/\sqrt{n} = 0.18/\sqrt{36} = 0.03}$. The sampling distribution of

$$z = \frac{\bar{x} - \mu_0}{\sigma_{\bar{x}}} = \frac{\bar{x} - 3}{0.03} \text{ is a standard normal distribution.}$$



- A value of $z = -1$ ($z = -3$) means that the value of \bar{x} is one (three) standard error below the hypothesized value of the mean.
- The lower tail area at $z = -3.00$ is 0.0013. Hence, the probability of obtaining a value of z that is three or more standard errors below the mean is 0.0013.
- The probability of obtaining a value of \bar{x} that is 3 or more standard errors below the hypothesized population mean $\mu_0 = 3$ is also 0.0013. Such a result is unlikely if the null hypothesis is true.
- For hypothesis tests about a population mean in the σ known case, we use the standard normal random variable z as a test statistic to determine whether \bar{x} deviates from the hypothesized value of μ enough to justify rejecting the null hypothesis.
- Test Statistic for Hypothesis Tests About a Population Mean: σ Known**

$$z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$

- The key question for a lower tail test is, How small must the test statistic z be before we choose to reject H_0 ? Two approaches: the p -value approach and the critical value approach.

p -value approach

1. **p -value:** A p -value is a probability that provides a measure of the evidence against H_0 provided by the sample.
 - (a) The p -value is used to determine whether H_0 should be rejected.
 - (b) A small p -value indicates the value of the test statistic is unusual given the assumption that H_0 is true.
 - (c) Smaller p -values indicate more evidence against H_0 .
2. The value of the test statistic is used to compute the p -value.
 - (a) For a lower tail test, the p -value is the probability of obtaining a value for the test statistic as small as or smaller than that provided by the sample.
 - (b) To compute the p -value for the lower tail test in the σ known case, we use the standard normal distribution to find the probability that z is less than or equal to the value of the test statistic.
 - (c) After computing the p -value, we must then decide whether it is small enough to reject the null hypothesis; as we will show, this decision involves comparing the p -value to the level of significance.

 **Question** (p427)

Suppose the sample of 36 Hilltop coffee cans provides a sample mean of $\bar{x} = 2.92$ pounds. Is $\bar{x} = 2.92$ small enough to cause us to reject H_0 ? Compute the p -value for the Hilltop Coffee lower tail test.

sol:

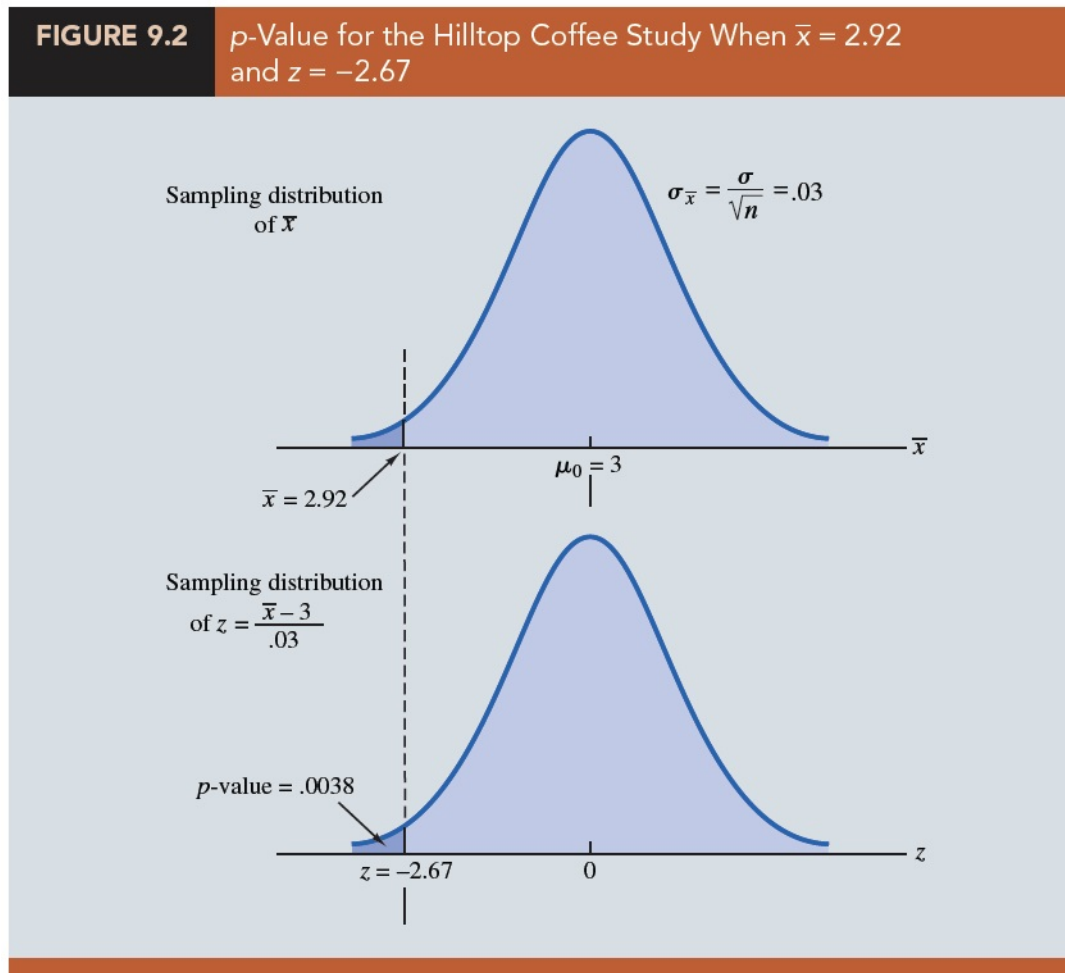
- Because this is a lower tail test, the p -value is the area under the standard normal curve for values of $z \leq$ the value of the test statistic.

– Using $\bar{x} = 2.92$, $\sigma = 0.18$, and $n = 36$, the value of the test statistic z :

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{2.92 - 3}{0.18/\sqrt{36}} = -2.67$$

– p -value $= P(z \leq -2.67) = 0.0038$.

– (Figure 9.2) $\bar{x} = 2.92$ corresponds to $z = -2.67$ and a p -value = 0.0038.



3. This p -value (0.0038) indicates a small probability of obtaining a sample mean of $\bar{x} = 2.92$ (and a test statistic of -2.67) or smaller when sampling from a population with $\mu = 3$.

4. This p -value does not provide much support for the null hypothesis, but is it small enough to cause us to reject H_0 ? The answer depends upon α for the test.
5. As noted previously, the director of the FTC's testing program selected a value of 0.01 for the level of significance means that the director is willing to tolerate a probability of 0.01 of rejecting H_0 when it is true as an equality ($\mu_0 = 3$).
6. The sample of 36 coffee cans in the Hilltop Coffee study resulted in a p -value = 0.0038, which means that the probability of obtaining a value of $\bar{x} = 2.92$ or less when H_0 is true as an equality is 0.0038.
7. Because $p\text{-value} = 0.0038 \leq \alpha = 0.01$, we reject H_0 . Therefore, we find sufficient statistical evidence to reject the null hypothesis at the 0.01 level of significance.
8. **Rejection Rule Using p -value.** For a level of significance α , the rejection rule using the p -value approach is:

$$\underline{\text{Reject } H_0 \text{ if } p\text{-value} \leq \alpha}$$

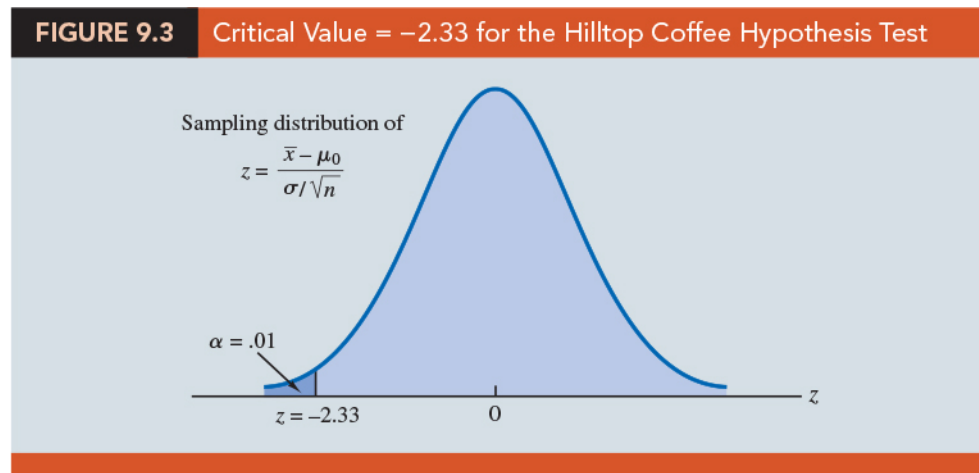
9. **Example** In the Hilltop Coffee test, the p -value of 0.0038 resulted in the rejection of H_0 . The observed p -value of 0.0038 means that we would reject H_0 for any value of $\alpha \geq 0.0038$. For this reason, the p -value is also called the observed level of significance.
10. Different decision makers may express different opinions concerning the cost of making a Type I error and may choose a different α .

Critical value approach

1. The critical value is the value of the test statistic that corresponds to an area of α in the lower tail of the sampling distribution of the test statistic.
2. The critical value is the largest value of the test statistic that will result in the rejection of the null hypothesis.
3. For a lower tail test, the critical value serves as a benchmark for determining whether the value of the test statistic is small enough to reject the null hypothesis.

4. **Example** the Hilltop Coffee example.

- (a) In the σ known case, the sampling distribution for the test statistic z is a standard normal distribution. Therefore, the critical value is the value of the test statistic that corresponds to an area of $\alpha = 0.01$ in the lower tail of a standard normal distribution.
- (b) (Figure 9.3) We find that $-z_{0.01} = -2.33$ provides an area of 0.01 in the lower tail, $P(z \leq -2.33) = 0.01$.
- (c) If the sample results in a value of the test statistic that is less than or equal to -2.33 , the corresponding p -value will be less than or equal to 0.01; in this case, we should reject H_0 .



- (d) Hence, for the Hilltop Coffee study the critical value rejection rule for a level of significance of 0.01 is

$$\text{Reject } H_0 \text{ if } \underline{z \leq -2.33}$$

- (e) In the Hilltop Coffee example, $\bar{x} = 2.92$ and the test statistic is $z = -2.67$. Because $z = -2.67 < -2.33$, we can reject H_0 and conclude that Hilltop Coffee is underfilling cans.

5. Rejection Rule for a Lower Tail Test: Critical Value Approach. We can generalize the rejection rule for the critical value approach to handle any level of significance. The rejection rule for a lower tail test follows.

$$\text{Reject } H_0 \text{ if } \underline{z \leq -z_\alpha}$$

where $-z_\alpha$ is the critical value; that is, the z value that provides an area of α in the lower tail of the standard normal distribution.

Summary

1. The p -value approach to hypothesis testing and the critical value approach will always lead to the same rejection decision.
2. The advantage of the p -value approach is that the p -value tells us how significant the results are (the observed level of significance).
3. If we use the critical value approach, we only know that the results are significant at the stated α .
4. We can use the same general approach to conduct an upper tail test. The test statistic z is still computed using equation (9.1). But, for an upper tail test, the p -value is the probability of obtaining a value for the test statistic as large as or larger than that provided by the sample.
5. To compute the p -value for the upper tail test in the σ known case, we must use the standard normal distribution to find the probability that z is greater than or equal to the value of the test statistic.

6. Computation of p -Values for One-Tailed Tests

- (a) Compute the value of the test statistic using equation (9.1):

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

- (b) *Lower tail test*: Using the standard normal distribution, compute the probability that z is less than or equal to the value of the test statistic (area in the lower tail).
- (c) *Upper tail test*: Using the standard normal distribution, compute the probability that z is greater than or equal to the value of the test statistic (area in the upper tail).

Two-tailed Test

1. The general form for a two-tailed test about a population mean:

$$\underline{H_0 : \mu = \mu_0, \quad H_a : \mu \neq \mu_0}$$

2. **Example** The U.S. Golf Association (USGA) establishes rules that manufacturers of golf equipment must meet if their products are to be acceptable for use in USGA events. MaxFlight Inc. uses a high-technology manufacturing process to produce golf balls with a mean driving distance of 295 yards. Sometimes, however, the process gets out of adjustment and produces golf balls with a mean driving distance different from 295 yards. When the mean distance falls below 295 yards, the company worries about losing sales because the golf balls do not provide as much distance as advertised. When the mean distance passes 295 yards, MaxFlight's golf balls may be rejected by the USGA for exceeding the overall distance standard concerning carry and roll. MaxFlight's quality control program involves taking periodic samples of 50 golf balls to monitor the manufacturing process. For each sample, a hypothesis test is conducted to determine whether the process has fallen out of adjustment.

- (a) We begin by assuming that the process is functioning correctly; that is, the golf balls being produced have a mean distance of 295 yards. This assumption establishes the null hypothesis. The alternative hypothesis is that the mean distance is not equal to 295 yards.

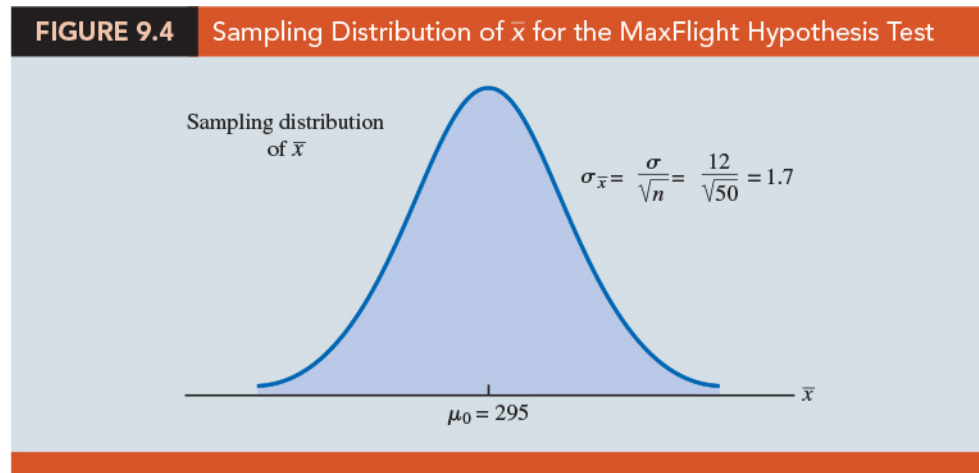
$$\underline{H_0 : \mu = 295, \quad H_a : \mu \neq 295} .$$

- (b) If the sample mean \bar{x} is significantly less than 295 yards or significantly greater than 295 yards, we will reject H_0 . In this case, corrective action will be taken to adjust the manufacturing process.
- (c) If \bar{x} does not deviate from the hypothesized mean $\mu_0 = 295$ by a significant amount, H_0 will not be rejected and no action will be taken to adjust the manufacturing process.
- (d) The quality control team selected $\alpha = 0.05$ as the level of significance for the test. Data from previous tests conducted when the process was known to be

in adjustment show that the population standard deviation can be assumed known with a value of $\sigma = 12$. Thus, with a sample size of $n = 50$, the standard error of \bar{x} is

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{12}{\sqrt{50}} = 1.7$$

- (e) Because the sample size is large, the central limit theorem allows us to conclude that the sampling distribution of \bar{x} can be approximated by a normal distribution.
- (f) (Figure 9.4) the sampling distribution of \bar{x} for the MaxFlight hypothesis test with a hypothesized population mean of $\mu_0 = 295$.



3. Suppose that a sample of 50 golf balls is selected and that the sample mean is $\bar{x} = 297.6$ yards. This sample mean provides support for the conclusion that the population mean is larger than 295 yards. Is this value of \bar{x} enough larger than 295 to cause us to reject H_0 at the 0.05 level of significance?

***p*-value approach**

- (Recall) the *p*-value is a probability used to determine whether the null hypothesis should be rejected.
- For a two-tailed test, values of the test statistic in either tail provide evidence against the null hypothesis.

3. For a two-tailed test, the p -value is the probability of obtaining a value for the test statistic as unlikely as or more unlikely than that provided by the sample.

4. **Example** the MaxFlight hypothesis test example.

(a) *Compute the value of the test statistic.* For the σ known case, the test statistic z is a standard normal random variable. Using equation (9.1) with $\bar{x} = 297.6$, the value of the test statistic is

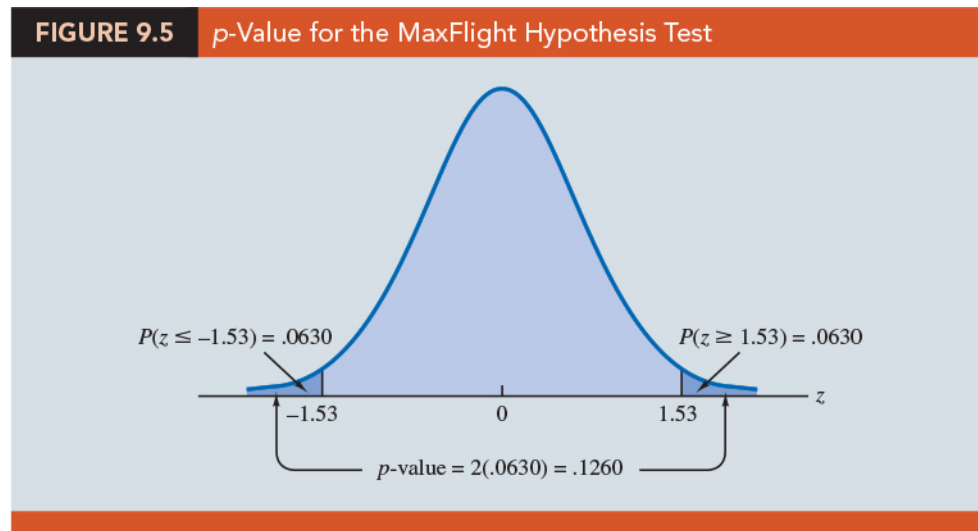
$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{297.6 - 295}{12/\sqrt{50}} = 1.53$$

(b) *Compute the p -value.* Find the probability of obtaining a value for the test statistic at least as unlikely as $z = 1.53$. Clearly values of $z \geq 1.53$ are at least as unlikely.

(c) But, because this is a two-tailed test, values of $z \leq -1.53$ are also at least as unlikely as the value of the test statistic provided by the sample.

(d) (Figure 9.5) the two-tailed p -value: $P(z \leq -1.53) + P(z \geq 1.53)$.

(e) Because the normal curve is symmetric, $P(z < 1.53) = 0.9370$. Thus, the upper tail area is $P(z \geq 1.53) = \underline{1.0000 - 0.9370 = 0.0630}$.



(f) The p -value for the MaxFlight two-tailed hypothesis test is

$$p\text{-value} = \underline{2(0.0630) = 0.1260}$$

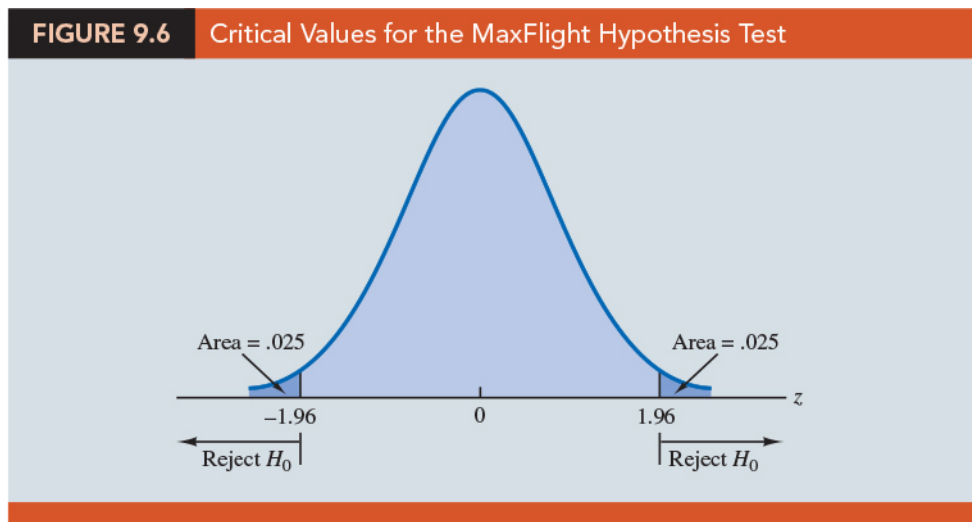
- (g) With a level of significance of $\alpha = 0.05$, we do not reject H_0 because the $p\text{-value} = 0.1260 > 0.05$. Because the null hypothesis is not rejected, no action will be taken to adjust the MaxFlight manufacturing process.

5. Computation of p -Values for Two-Tailed Tests.

- (a) Compute the value of the test statistic using equation (9.1).
- (b) If the value of the test statistic is in the upper tail, compute the probability that z is greater than or equal to the value of the test statistic (the upper tail area).
- (c) If the value of the test statistic is in the lower tail, compute the probability that z is less than or equal to the value of the test statistic (the lower tail area).
- (d) Double the probability (or tail area) from step (b) or (c) to obtain the p -value.

Critical value approach

1. (Figure 9.6) the critical values for the test will occur in both the lower and upper tails of the standard normal distribution. With a level of significance of $\alpha = 0.05$, the area in each tail corresponding to the critical values is $\alpha/2 = 0.05/2 = 0.025$.



2. The critical values for the test statistic are $-z_{0.025} = -1.96$ and $z_{0.025} = 1.96$.

3. The two-tailed rejection rule is

$$\text{Reject } H_0 \text{ if } z \leq -1.96 \text{ or if } z \geq 1.96$$

4. Because the value of the test statistic for the MaxFlight study is $z = 1.53$, the statistical evidence will not permit us to reject the null hypothesis at the 0.05 level of significance.

Summary and Practical Advice

1. Summary of the hypothesis testing procedures about a population mean for the σ known case. Note that μ_0 is the hypothesized value of the population mean.

TABLE 9.2 Summary of Hypothesis Tests About a Population Mean: σ Known Case			
	Lower Tail Test	Upper Tail Test	Two-Tailed Test
Hypotheses	$H_0: \mu \geq \mu_0$ $H_a: \mu < \mu_0$	$H_0: \mu \leq \mu_0$ $H_a: \mu > \mu_0$	$H_0: \mu = \mu_0$ $H_a: \mu \neq \mu_0$
Test Statistic	$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$	$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$	$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$
Rejection Rule: p-Value Approach	Reject H_0 if $p\text{-value} \leq \alpha$	Reject H_0 if $p\text{-value} \leq \alpha$	Reject H_0 if $p\text{-value} \leq \alpha$
Rejection Rule: Critical Value Approach	Reject H_0 if $z \leq -z_\alpha$	Reject H_0 if $z \geq z_\alpha$	Reject H_0 if $z \leq -z_{\alpha/2}$ or if $z \geq z_{\alpha/2}$

2. Steps of Hypothesis Testing

Step 1. Develop the null and alternative hypotheses (H_0, H_a).

Step 2. Specify the level of significance (α).

Step 3. Collect the sample data ($\mathbf{x} = \{x_1, x_2, \dots, x_n\}$) and compute the value of the test statistic ($T(\mathbf{x}) = z$).

p-value Approach

Step 4. Use the value of the test statistic to compute the p-value.

Step 5. Reject H_0 if the $p\text{-value} \leq \alpha$.

Step 6. Interpret the statistical conclusion in the context of the application.

Critical Value Approach

Step 4. Use α to determine the critical value (z_α or $z_{\alpha/2}$) and the rejection rule.

Step 5. Use the value of the test statistic and the rejection rule to determine whether to reject H_0 .

Step 6. Interpret the statistical conclusion in the context of the application.

3. Practical advice about the sample size for hypothesis tests is similar to the advice we provided about the sample size for interval estimation in Chapter 8.

(a) In most applications, a sample size of $n \geq 30$ is adequate when using the hypothesis testing procedure described in this section.

(b) If the population is normally distributed, the hypothesis testing procedure that we described is exact and can be used for any sample size.

(c) If the population is not normally distributed but is at least roughly symmetric, sample sizes as small as 15 can be expected to provide acceptable results.

Relationship Between Interval Estimation and Hypothesis Testing

1. (Recall, Chapter 8) For the σ known case, the $(1-\alpha)\%$ confidence interval estimate of a population mean is given by

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}.$$

2. (Recall, Chapter 9) a two-tailed hypothesis test about a population mean:

$$H_0 : \mu = \mu_0, \quad H_a : \mu \neq \mu_0$$

where μ_0 is the hypothesized value for the population mean.

3. Constructing a $100(1-\alpha)\%$ confidence interval for the population mean: $100(1-\alpha)\%$ of the confidence intervals generated will contain the population mean and $100\alpha\%$ of the confidence intervals generated will not contain the population mean.
4. If we reject H_0 whenever the confidence interval does not contain μ_0 , we will be rejecting H_0 when it is true ($\mu = \mu_0$) with probability α .
5. Recall that α is the probability of rejecting the null hypothesis when it is true.
6. So constructing a $100(1-\alpha)\%$ confidence interval and rejecting H_0 whenever the interval does not contain μ_0 is equivalent to conducting a two-tailed hypothesis test with α as the level of significance.
7. A Confidence Interval Approach to Testing a Hypothesis of the Form:

$$H_0 : \mu = \mu_0, \quad H_a : \mu \neq \mu_0$$

- (a) Select a simple random sample from the population and use the value of the sample mean \bar{x} to develop the confidence interval for the population mean μ .

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

- (b) If the confidence interval contains the hypothesized value μ_0 , do not reject H_0 . Otherwise, reject H_0 .
8. Note that this discussion and example pertain to two-tailed hypothesis tests about a population mean. However, the same confidence interval and two-tailed hypothesis testing relationship exists for other population parameters.
9. The relationship can also be extended to one-tailed tests about population parameters. Doing so, however, requires the development of one-sided confidence intervals, which are rarely used in practice.

 Question (p435)

The MaxFlight hypothesis test takes the following form:

$$H_0 : \mu = 295, \quad H_a : \mu \neq 295.$$

Conducting the MaxFlight hypothesis test with a level of significance of $\alpha = 0.05$ using the confidence interval approach.

sol:

- We sampled $n = 50$ golf balls and found a sample mean distance of $\bar{x} = 297.6$ yards. Recall that the population standard deviation is $\sigma = 12$.

- The 95% confidence interval estimate of the population mean is

$$\begin{aligned} & \frac{\bar{x} \pm z_{0.025} \frac{\sigma}{\sqrt{n}}}{297.6 \pm 1.96 \frac{12}{\sqrt{50}}} \\ & 297.6 \pm 3.3 \quad \text{or} \quad (294.3, 300.9). \end{aligned}$$

- With 95% confidence, the mean distance for the population of golf balls is between 294.3 and 300.9 yards.
- Because the interval contains the hypothesized value for the population mean, $\mu_0 = 295$, the hypothesis testing conclusion is that the null hypothesis, $H_0 : \mu = 295$, cannot be rejected.

9.4 Population Mean: σ Unknown

1. To conduct a hypothesis test about a population mean for the σ unknown case, the sample mean \bar{x} is used as an estimate of μ and the sample standard deviation s is used as an estimate of σ .

2. (Recall) For the σ known case, the sampling distribution of the test statistic has a standard normal distribution. For the σ unknown case, however, the sampling distribution of the test statistic follows the t distribution; it has slightly more variability because the sample is used to develop estimates of both μ and σ .
3. **Test Statistic for Hypothesis Tests about a Population Mean: σ Unknown**
the test statistic has a t distribution with $n-1$ degrees of freedom:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \quad (9.2)$$

4. The t distribution is based on an assumption that the population from which we are sampling has a normal distribution. However, research shows that this assumption can be relaxed considerably when the sample size is large enough.

One-tailed Test

1. **Example** A business travel magazine wants to classify transatlantic gateway airports according to the mean rating for the population of business travelers. A rating scale with a low score of 0 and a high score of 10 will be used, and airports with a population mean rating greater than 7 will be designated as superior service airports. The magazine staff surveyed a sample of 60 business travelers at each airport to obtain the ratings data. The sample for London's Heathrow Airport provided a sample mean rating of $\bar{x} = 7.25$ and a sample standard deviation of $s = 1.052$. Do the data indicate that Heathrow should be designated as a superior service airport?
- (a) We want to develop a hypothesis test for which the decision to reject H_0 will lead to the conclusion that the population mean rating for the Heathrow Airport is greater than 7.
- (b) The null and alternative hypotheses for this upper tail test:

$$H_0 : \mu \leq 7, \quad H_a : \mu > 7$$

- (c) Use $\alpha = 0.05$, with $\bar{x} = 7.25$, $\mu_0 = 7$, $s = 1.052$, and $n = 60$, the value of the test statistic:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{7.25 - 7}{1.052/\sqrt{60}} = 1.84$$

- (d) The sampling distribution of t has $n-1 = 60-1 = 59$ degrees of freedom. Because the test is an upper tail test, the p -value is $P(t \geq 1.84)$, that is, the upper tail area corresponding to the value of the test statistic.
- (e) (Table 2 in Appendix B) the t distribution with 59 degrees of freedom provides the following information.

Area in Upper Tail	0.20	0.10	0.05	0.025	0.01	0.005
t -Value (59 df)	0.848	1.296	1.671	2.001	2.391	2.662

- i. We see that $t = 1.84$ is between 1.671 and 2.001. The values in the "Area in Upper Tail" row show that the p -value must be less than 0.05 and greater than 0.025.
- ii. With a level of significance of $\alpha = 0.05$, this placement is all we need to know to make the decision to reject the null hypothesis and conclude that Heathrow should be classified as a superior service airport.
- (f) (Using software) $t = 1.84$ provides the upper tail p -value of 0.0354 for the Heathrow Airport hypothesis test. With $0.0354 < 0.05$, we reject the null hypothesis and conclude that Heathrow should be classified as a superior service airport.
- (g) The critical value corresponding to an area of $\alpha = 0.05$ in the upper tail of a t distribution with 59 degrees of freedom is $t_{59;0.05} = 1.671$.
- (h) The rejection rule using the critical value approach is to reject H_0 if $t \geq 1.671$. Because $t = 1.84 > 1.671$, H_0 is rejected. Heathrow should be classified as a superior service airport.

Two-tailed Test

- Example** Consider the hypothesis testing situation facing Holiday Toys. The company manufactures and distributes its products through more than 1000 retail outlets. In planning production levels for the coming winter season, Holiday must decide how many units of each product to produce prior to knowing the actual demand at the retail level. For this year's most important new toy, Holiday's marketing director is expecting demand to average 40 units per retail outlet. Prior to making the final production decision based upon this estimate, Holiday decided

to survey a sample of 25 retailers in order to develop more information about the demand for the new product. Each retailer was provided with information about the features of the new toy along with the cost and the suggested selling price. Then each retailer was asked to specify an anticipated order quantity. With μ denoting the population mean order quantity per retail outlet, the sample data will be used to conduct the following two-tailed hypothesis test:

$$\underline{H_0 : \mu = 40, \quad H_a : \mu \neq 40}$$

- (a) If H_0 cannot be rejected, Holiday will continue its production planning based on the marketing director's estimate that the population mean order quantity per retail outlet will be $\mu = 40$ units.
- (b) If H_0 is rejected, Holiday will immediately reevaluate its production plan for the product.
- (c) A two-tailed hypothesis test is used because Holiday wants to reevaluate the production plan if the population mean quantity per retail outlet is less than anticipated or greater than anticipated.
- (d) Because no historical data are available (it's a new product), the population mean μ and the population standard deviation must both be estimated using \bar{x} and s from the sample data.
- (e) The sample of 25 retailers provided a mean of $\bar{x} = 37.4$ and a standard deviation of $s = 11.79$ units.
- (f) (*Check on the form of the population distribution*). The histogram of the sample data showed no evidence of skewness or any extreme outliers, so the analyst concluded that the use of the t distribution with $n-1 = 24$ degrees of freedom was appropriate.
- (g) Using equation (9.2) with $\bar{x} = 37.4$, $\mu_0 = 40$, $s = 11.79$, and $n = 25$, the value of the test statistic is

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{37.4 - 40}{11.79/\sqrt{25}} = -1.10 .$$

- (h) The t distribution table only contains positive t values. Because the t distribution is symmetric, however, the upper tail area at $t = 1.10$ is the same as the lower tail area at $t = -1.10$.

(i) (Table 2 in Appendix B)

Area in Upper Tail	0.20	0.10	0.05	0.025	0.01	0.005
t-Value (24 <i>df</i>)	0.857	1.318	1.711	2.064	2.492	2.797

(j) We see that $t = 1.10$ is between 0.857 and 1.318. From the "Area in Upper Tail" row, we see that the area in the upper tail at $t = 1.10$ is between 0.20 and 0.10.

(k) When we double these amounts, we see that the p -value must be between 0.40 and 0.20. With a level of significance of $\alpha = 0.05$, we now know that the p -value is greater than α . Therefore, H_0 cannot be rejected. Sufficient evidence is not available to conclude that Holiday should change its production plan for the coming season.

(l) (Software) The p -value obtained is 0.2822. With a level of significance of $\alpha = 0.05$, we cannot reject H_0 because $0.2822 > 0.05$.

(m) With $\alpha = 0.05$ and the t distribution with 24 degrees of freedom, $-t_{24,0.025} = -2.064$ and $t_{24,0.025} = 2.064$ are the critical values for the two-tailed test. The rejection rule using the test statistic is

$$\underline{\text{Reject } H_0 \text{ if } t \leq -2.064 \text{ or if } t \geq 2.064}.$$

(n) Based on the test statistic $t = -1.10$, H_0 cannot be rejected. This result indicates that Holiday should continue its production planning for the coming season based on the expectation that $\mu = 40$.

Summary and Practical Advice

- (Table 9.3) A summary of the hypothesis testing procedures about a population mean for the σ unknown case.

TABLE 9.3 Summary of Hypothesis Tests About a Population Mean: σ Unknown Case

	Lower Tail Test	Upper Tail Test	Two-Tailed Test
Hypotheses	$H_0: \mu \geq \mu_0$ $H_a: \mu < \mu_0$	$H_0: \mu \leq \mu_0$ $H_a: \mu > \mu_0$	$H_0: \mu = \mu_0$ $H_a: \mu \neq \mu_0$
Test Statistic	$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$	$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$	$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$
Rejection Rule: p-Value Approach	Reject H_0 if $p\text{-value} \leq \alpha$	Reject H_0 if $p\text{-value} \leq \alpha$	Reject H_0 if $p\text{-value} \leq \alpha$
Rejection Rule: Critical Value Approach	Reject H_0 if $t \leq -t_\alpha$	Reject H_0 if $t \geq t_\alpha$	Reject H_0 if $t \leq -t_{\alpha/2}$ or if $t \geq t_{\alpha/2}$

2. The applicability of the hypothesis testing procedures of this section is dependent on the distribution of the population being sampled from and the sample size.

9.5 Population Proportion

1. Using p_0 to denote the hypothesized value for the population proportion, the three forms for a hypothesis test about a population proportion:

lower tail test	upper tail test	two-tailed test
$H_0 : p \geq p_0$	$H_0 : p \leq p_0$	$H_0 : p = p_0$
$H_a : p < p_0$	$H_a : p > p_0$	$H_a : p \neq p_0$

2. Hypothesis tests about a population proportion are based on the difference between the sample proportion \bar{p} and the hypothesized population proportion p_0 .
3. The sampling distribution of \bar{p} , the point estimator of the population parameter p , is the basis for developing the test statistic.

4. When the null hypothesis is true as an equality, the expected value of \bar{p} equals the hypothesized value p_0 ; that is, $E(\bar{p}) = p_0$. The standard error of \bar{p} is given by

$$\sigma_{\bar{p}} = \sqrt{\frac{p_0(1-p_0)}{n}}$$

5. (Recall, Chapter 7) we said that if $np \geq 5$ and $n(1-p) \geq 5$, the sampling distribution of \bar{p} can be approximated by a normal distribution. Under these conditions, which usually apply in practice, the quantity

$$z = \frac{\bar{p} - p_0}{\sigma_{\bar{p}}} \quad (9.3)$$

has a standard normal probability distribution.

6. Test Statistic for Hypothesis Tests About a Population Proportion

With $\sigma_{\bar{p}} = \sqrt{p_0(1-p_0)/n}$, the standard normal random variable z is the test statistic used to conduct hypothesis tests about a population proportion.

$$z = \frac{\bar{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

7. **Example** (Pine Creek golf course example). Over the past year, 20% of the players at Pine Creek were women. In an effort to increase the proportion of women players, Pine Creek implemented a special promotion designed to attract women golfers. One month after the promotion was implemented, the course manager requested a statistical study to determine whether the proportion of women players at Pine Creek had increased.

- (a) Because the objective of the study is to determine whether the proportion of women golfers increased, an upper tail test with $H_a : p > 0.20$ is appropriate:

$$H_0 : p \leq 0.20, \quad H_a : p > 0.20$$

- (b) If H_0 can be rejected, the test results will give statistical support for the conclusion that the proportion of women golfers increased and the promotion was beneficial.

- (c) The course manager specified that a level of significance of $\alpha = 0.05$ be used in carrying out this hypothesis test.
- (d) The next step of the hypothesis testing procedure is to select a sample and compute the value of an appropriate test statistic. Suppose a random sample of $n = 400$ players was selected, and that $x = 100$ of the players were women. The proportion of women golfers in the sample is

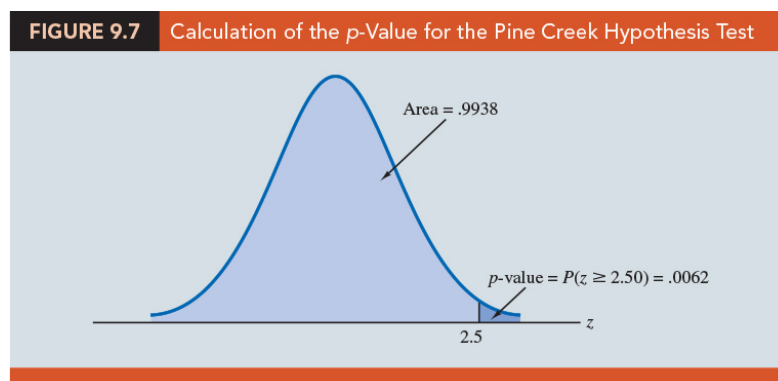
$$\bar{p} = \frac{100}{400} = 0.25$$

Using equation (9.4), the value of the test statistic is

$$z = \frac{\bar{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.25 - 0.20}{\sqrt{\frac{0.20(1-0.20)}{400}}} = \frac{0.05}{0.02} = 2.50.$$

(e) *The p-value approach.*

- i. The p -value is the probability that z is greater than or equal to $z = 2.50$. $P(z \geq 2.50) = 0.9938$, the p -value for the Pine Creek test is $1.0000 - 0.9938 = 0.0062$.
- ii. (Figure 9.7) Recall that the course manager specified a level of significance of $\alpha = 0.05$. A $p\text{-value} = 0.0062 < 0.05$ gives sufficient statistical evidence to reject H_0 at the 0.05 level of significance.
- iii. The test provides statistical support for the conclusion that the special promotion increased the proportion of women players at the Pine Creek golf course.



- (f) *The critical value approach.* The critical value corresponding to an area of 0.05 in the upper tail of a normal probability distribution is $z_{0.05} = 1.645$.

Thus, the rejection rule using the critical value approach is to reject H_0 if $z \geq 1.645$. Because $z = 2.50 > 1.645$, H_0 is rejected.

- (g) The p -value approach provides more information. With a p -value = 0.0062, the null hypothesis would be rejected for any level of significance greater than or equal to 0.0062.

Summary

- The procedure used to conduct a hypothesis test about a population proportion is similar to the procedure used to conduct a hypothesis test about a population mean.
- Although we only illustrated how to conduct a hypothesis test about a population proportion for an upper tail test, similar procedures can be used for lower tail and two-tailed tests.
- (Table 9.4) A summary of the hypothesis tests about a population proportion. We assume that $np \geq 5$ and $n(1-p) \geq 5$; thus the normal probability distribution can be used to approximate the sampling distribution of \bar{p} .

TABLE 9.4 Summary of Hypothesis Tests About a Population Proportion			
	Lower Tail Test	Upper Tail Test	Two-Tailed Test
Hypotheses	$H_0: p \geq p_0$ $H_a: p < p_0$	$H_0: p \leq p_0$ $H_a: p > p_0$	$H_0: p = p_0$ $H_a: p \neq p_0$
Test Statistic	$z = \frac{\bar{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$	$z = \frac{\bar{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$	$z = \frac{\bar{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$
Rejection Rule: p-Value Approach	Reject H_0 if $p\text{-value} \leq \alpha$	Reject H_0 if $p\text{-value} \leq \alpha$	Reject H_0 if $p\text{-value} \leq \alpha$
Rejection Rule: Critical Value Approach	Reject H_0 if $z \leq -z_\alpha$	Reject H_0 if $z \geq z_\alpha$	Reject H_0 if $z \leq -z_{\alpha/2}$ or if $z \geq z_{\alpha/2}$

9.6 Hypothesis Testing and Decision Making

- The hypothesis testing applications are considered as significance tests :
 - formulate the null and alternative hypotheses, H_0, H_a .
 - specify the level of significance, α .
 - select a sample, $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$.
 - compute the value of a test statistic, $T(\mathbf{x})$.
 - compute the associated p -value.
 - compare the p -value to α .
 - conclude "reject H_0 " and declare the results significant if $p\text{-value} \leq \alpha$; otherwise, we made the conclusion "do not reject H_0 ."
- With a significance test, we control the probability of making the Type I error, but not the Type II error. Thus, we recommended the conclusion "do not reject H_0 " rather than "accept H_0 " because the latter puts us at risk of making the Type II error of accepting H_0 when it is false.
- With the conclusion "do not reject H_0 ," the statistical evidence is considered inconclusive and is usually an indication to postpone a decision or action until further research and testing can be undertaken.
- If the purpose of a hypothesis test is to make a decision when H_0 is true and a different decision when H_a is true, the decision maker may want to, and in some cases be forced to, take action with both the conclusion do not reject H_0 and the conclusion reject H_0 . If this situation occurs, statisticians generally recommend controlling the probability of making a Type II error .
- With the probabilities of both the Type I and Type II error controlled, the conclusion from the hypothesis test is either to accept H_0 or reject H_0 .
- Example (lot-acceptance example) A quality control manager must decide to accept a shipment of batteries from a supplier or to return the shipment because of poor quality.

- (a) Assume that design specifications require batteries from the supplier to have a mean useful life of at least 120 hours. To evaluate the quality of an incoming shipment, a sample of 36 batteries will be selected and tested.
- (b) On the basis of the sample, a decision must be made to accept the shipment of batteries or to return it to the supplier because of poor quality.
- (c) Let μ denote the mean number of hours of useful life for batteries in the shipment. The null and alternative hypotheses about the population mean:

$$\underline{H_0 : \mu \geq 120, \quad H_a : \mu < 120}$$

- i. If H_0 is rejected, the alternative hypothesis is concluded to be true. This conclusion indicates that the appropriate action is to return the shipment to the supplier.
- ii. If H_0 is not rejected, the decision maker must still determine what action should be taken. Thus, without directly concluding that H_0 is true, but merely by not rejecting it, the decision maker will have made the decision to accept the shipment as being of satisfactory quality.
- (d) In such decision-making situations, it is recommended that the hypothesis testing procedure be extended to control the probability of making a Type II error.
7. Because a decision will be made and action taken when we do not reject H_0 , knowledge of the probability of making a Type II error will be helpful.

9.7 Calculating The Probability of Type II Errors

1. **Example** (lot-acceptance example) The null and alternative hypotheses about the mean number of hours of useful life for a shipment of batteries:

$$H_0 : \mu \geq 120, \quad H_a : \mu < 120.$$

- (a) If H_0 is rejected, the decision will be to return the shipment to the supplier because the mean hours of useful life are less than the specified 120 hours.
- (b) If H_0 is not rejected, the decision will be to accept the shipment.

2. Suppose a level of significance of $\alpha = 0.05$, the test statistic in the σ known case:

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{x} - 120}{\sigma/\sqrt{n}}$$

3. The rejection rule for the lower tail test:

$$\text{Reject } H_0 \text{ if } z \leq -1.645$$

4. Suppose a sample of $n = 36$ batteries will be selected and based upon previous testing the population standard deviation can be assumed known with a value of $\sigma = 12$ hours.

5. The rejection rule indicates that we will reject H_0 if

$$z = \frac{\bar{x} - 120}{12/\sqrt{36}} \leq -z_{0.05} = -1.645$$

6. Solving for \bar{x} in the preceding expression indicates that we will reject H_0 if

$$\bar{x} \leq 120 - 1.645 \left(\frac{12}{\sqrt{36}} \right) = 116.71$$

7. Rejecting H_0 when $\bar{x} \leq 116.71$ means that we will make the decision to accept the shipment whenever $\bar{x} > 116.71$.

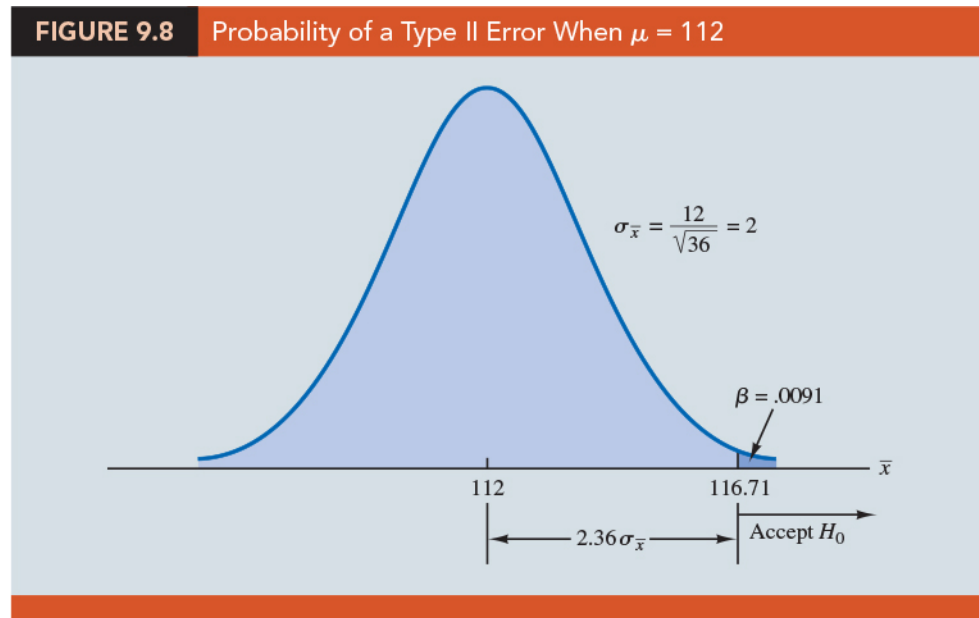
8. Compute probabilities associated with making a Type II error.

- (a) (Recall) we make a Type II error whenever the true shipment mean is less than 120 hours and we make the decision to accept $H_0 : \mu \geq 120$.
- (b) Hence, to compute the probability of making a Type II error, we must select a value of μ less than 120 hours.
- (c) For example, suppose the shipment is considered to be of poor quality if the batteries have a mean life of $\mu = 112$ hours.

- (d) If $\mu = 112$ is really true, what is the probability of accepting $H_0 : \mu \geq 120$ and hence committing a Type II error?

$$P(\bar{x} \geq 116.71), \text{ when } \mu = 112$$

- (e) (Figure 9.8) the sampling distribution of \bar{x} when the mean is $\mu = 112$. The shaded area in the upper tail gives the probability of obtaining $\bar{x} > 116.71$.



- (f) Using the standard normal distribution, we see that at $\bar{x} = 116.71$

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{116.71 - 112}{12/\sqrt{36}} = 2.36$$

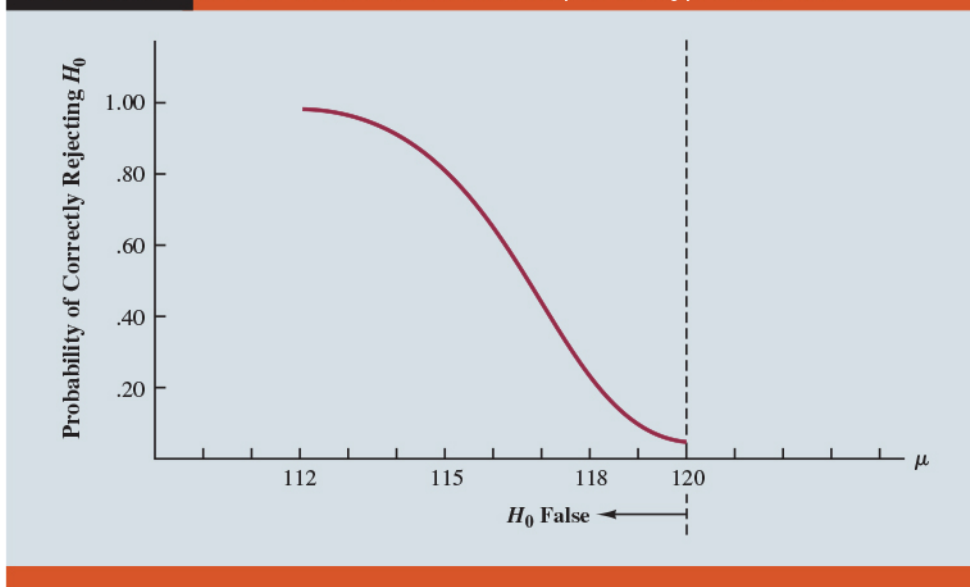
- (g) The probability of making a Type II error when $\mu = 112$ is $\beta := P(z \geq 2.36) = 0.0091$.
- (h) Therefore, we can conclude that if the mean of the population is 112 hours, the probability of making a Type II error is only 0.0091.
- (i) We can repeat these calculations for other values of μ less than 120.

9. (Table 9.5) we show the probability of making a Type II error for a variety of values of μ less than 120. Note that as μ increases toward 120, the probability of making a Type II error increases toward an upper bound of 0.95. However, as μ decreases to values farther below 120, the probability of making a Type II error diminishes.

TABLE 9.5 Probability of Making a Type II Error for the Lot-Acceptance Hypothesis Test

Value of μ	$z = \frac{116.71 - \mu}{12/\sqrt{36}}$	Probability of a Type II Error (β)	Power ($1 - \beta$)
112	2.36	.0091	.9909
114	1.36	.0869	.9131
115	.86	.1949	.8051
116.71	.00	.5000	.5000
117	-.15	.5596	.4404
118	-.65	.7422	.2578
119.999	-1.645	.9500	.0500

10. When the true population mean μ is close to (far below) the null hypothesis value of $\mu = 120$, the probability is high (low) that we will make a Type II error.
11. For any particular value of μ , the power is $1 - \beta$; that is, the probability of correctly rejecting H_0 is 1 minus the probability of making a Type II error.
12. (Figure 9.9) *Power curve*: the power associated with each value of μ :

FIGURE 9.9 Power Curve for the Lot-Acceptance Hypothesis Test

- (a) Note that the power curve extends over the values of μ for which the H_0 is false.

- (b) The height of the power curve at any value of μ indicates the probability of correctly rejecting H_0 when H_0 is false.
13. **The step-by-step procedure to compute the probability of making a Type II error in hypothesis tests about a population mean**
- Formulate the null and alternative hypotheses.
 - Use the level of significance α and the critical value approach to determine the critical value and the rejection rule for the test.
 - Use the rejection rule to solve for the value of the sample mean corresponding to the critical value of the test statistic.
 - Use the results from step (c) to state the values of the sample mean that lead to the acceptance of H_0 . These values define the acceptance region for the test.
 - Use the sampling distribution of \bar{x} for a value of μ satisfying H_a , and the acceptance region from step (d), to compute the probability that the sample mean will be in the acceptance region.
 - This probability is the probability of making a Type II error at the chosen value of μ .

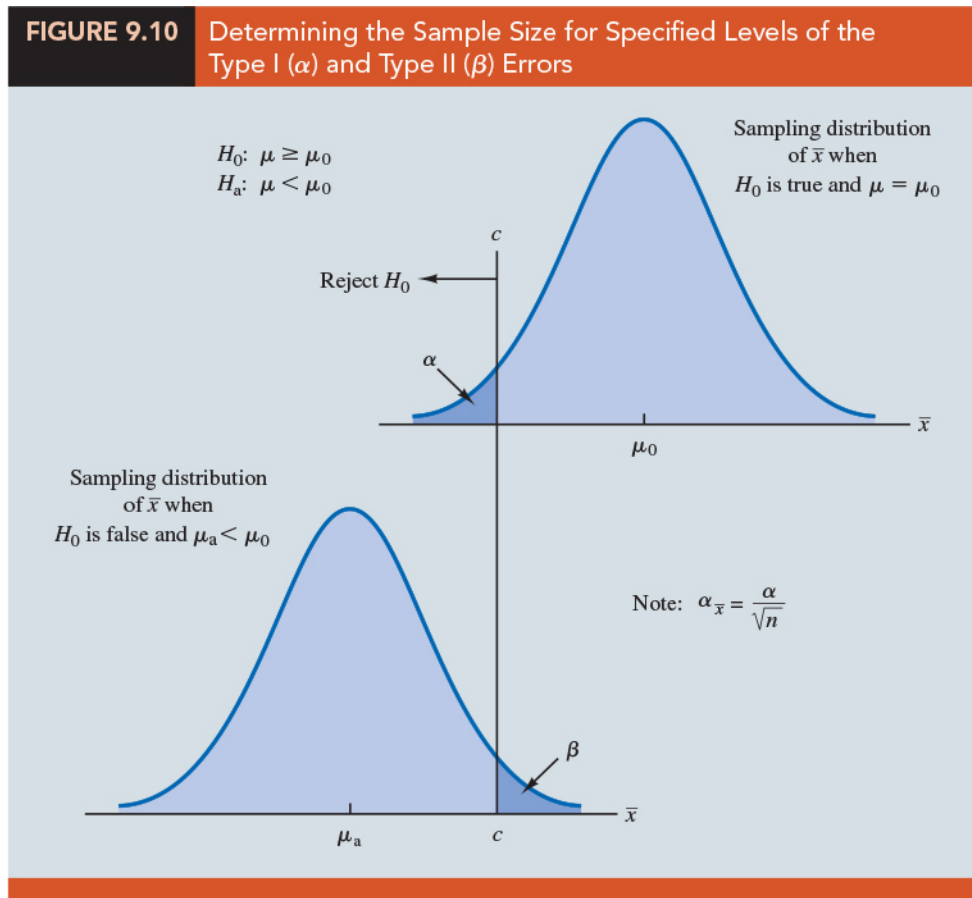
9.8 Determining The Sample Size for a Hypothesis Test about a Population Mean

- Assume that a hypothesis test is to be conducted about the value of a population mean. The level of significance specified by the user determines the probability of making a Type I error for the test. By controlling the sample size, the user can also control the probability of making a Type II error.

2. Let us show how a sample size can be determined for the following lower tail test about a population mean.

$$\underline{H_0 : \mu \geq \mu_0, \quad H_a : \mu < \mu_0}$$

3. (Figure 9.10) The upper panel is the sampling distribution of \bar{x} when H_0 is true with $\mu = \mu_0$.



4. For a lower tail test, the critical value of the test statistic is denoted $-z_\alpha$. In the upper panel of the figure the vertical line, labeled c , is the corresponding value of \bar{x} .
5. If we reject H_0 when $\bar{x} \leq c$, the probability of a Type I error will be α : $P(\bar{x} \leq c \mid H_0 T) = \alpha$.
6. With z_α representing the z value corresponding to an area of α in the upper tail of the standard normal distribution, we compute c using the following formula:

$$c = \underline{\mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}}}$$

7. (Figure 9.10) The lower panel is the sampling distribution of \bar{x} when the alternative hypothesis is true with $\mu = \mu_a < \mu_0$. The shaded region shows β , the probability of a Type II error that the decision maker will be exposed to if the null hypothesis is accepted when $\bar{x} > c$.
8. With z_β representing the z value corresponding to an area of β in the upper tail of the standard normal distribution, we compute c using the following formula:

$$c = \mu_a + z_\beta \frac{\sigma}{\sqrt{n}} \quad (9.6)$$

9. Now what we want to do is to select a value for c so that when we reject H_0 and accept H_a , the probability of a Type I error is equal to the chosen value of α and the probability of a Type II error is equal to the chosen value of β . Therefore, both equations (9.5) and (9.6) must provide the same value for c :

$$\begin{aligned} \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} &= \mu_a + z_\beta \frac{\sigma}{\sqrt{n}} \\ \mu_0 - \mu_a &= z_\alpha \frac{\sigma}{\sqrt{n}} + z_\beta \frac{\sigma}{\sqrt{n}} \\ \mu_0 - \mu_a &= \frac{(z_\alpha + z_\beta)\sigma}{\sqrt{n}} \\ \sqrt{n} &= \frac{(z_\alpha + z_\beta)\sigma}{(\mu_0 - \mu_a)} \end{aligned}$$

10. Sample Size for a One-Tailed Hypothesis Test About a Population Mean

$$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{(\mu_0 - \mu_a)^2} \quad (9.7)$$

where

- $z_\alpha = z$ value providing an area of α in the upper tail of a standard normal distribution.
- $z_\beta = z$ value providing an area of β in the upper tail of a standard normal distribution.
- $\sigma =$ the population standard deviation.
- $\mu_0 =$ the value of the population mean in the null hypothesis.
- $\mu_a =$ the value of the population mean used for the Type II error.

Note: In a two-tailed hypothesis test, use (9.7) with $z_{\alpha/2}$ replacing z_{α} .

11. **Example** lot-acceptance example

(a) The design specification for the shipment of batteries indicated a mean useful life of at least 120 hours for the batteries. Shipments were rejected if $H_0 : \mu \geq 120$ was rejected.

(b) Let us assume that the quality control manager makes the following statements about the allowable probabilities for the Type I and Type II errors.

i. *Type I error statement:* If the mean life of the batteries in the shipment is $\mu = 120$, I am willing to risk an $\alpha = 0.05$ probability of rejecting the shipment.

ii. *Type II error statement:* If the mean life of the batteries in the shipment is 5 hours under the specification (i.e., $\mu = 115$), I am willing to risk a $\beta = 0.10$ probability of accepting the shipment.

iii. These statements are based on the judgment of the manager. Someone else might specify different restrictions on the probabilities. However, statements about the allowable probabilities of both errors must be made before the sample size can be determined.

(c) In the example, $\alpha = 0.05$ and $\beta = 0.10$. Using the standard normal probability distribution, we have $z_{0.05} = 1.645$ and $z_{0.10} = 1.28$. From the statements about the error probabilities, we note that $\mu_0 = 120$ and $\mu_a = 115$. Finally, the population standard deviation was assumed known at $\sigma = 12$.

(d) The recommended sample size for the lot-acceptance example is

$$n = \frac{(z_{\alpha} + z_{\beta})^2 \sigma^2}{(\mu_0 - \mu_a)^2} = \frac{(1.645 + 1.28)^2 (12)^2}{(120 - 115)^2} = 49.3$$

Rounding up, we recommend a sample size of 50.

(e) Because both the Type I and Type II error probabilities have been **controlled** at allowable levels with $n = 50$, the quality control manager is now justified in using the accept H_0 and reject H_0 statements for the hypothesis test.

12. We can make three observations about the relationship among α , β , and the sample size n .

- (a) Once two of the three values are known, the other can be computed.
- (b) For a given α , increasing n will reduce β .
- (c) For a given n , decreasing α will increase β , whereas increasing α will decrease β .
13. The third observation should be kept in mind when the probability of a Type II error is not being controlled. It suggests that one should not choose unnecessarily small values for the level of significance α .
14. For a given sample size, choosing a smaller level of significance means more exposure to a Type II error. Inexperienced users of hypothesis testing often think that smaller values of α are always better. They are better if we are concerned only about making a Type I error. However, smaller values of α have the disadvantage of increasing the probability of making a Type II error.

9.9 Big Data And Hypothesis Testing*

☺ EXERCISES

9.1 : 2, 3

9.2 : 6, 7

9.3 : 9, 11, 15, 18, 22

9.4 : 23, 24, 27, 33

9.5 : 35, 36, 43, 44

9.7 : 46, 50, 53

9.8 : 55, 59

SUP : 69, 79, 83

“堅持做對的事，永遠不會錯。”

“You are never wrong to do the right thing.”

— 高年級實習生 (*The intern*, 2015)

統計學 (二)

Anderson's Statistics for Business & Economics (14/E)

Chapter 10: Inference About Means and Proportions with Two Populations

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Overview

1. Discuss the statistical inference (interval estimates and hypothesis tests) for two population means (three situations: population standard deviations known, unknown; match samples) and the two population proportions.
2. *Examples:*
 - (a) Develop an interval estimate of the difference between the mean starting salary for a population of men and the mean starting salary for a population of women.
 - (b) Conduct a hypothesis test to determine whether any difference is present between the proportion of defective parts in a population of parts produced by supplier *A* and the proportion of defective parts in a population of parts produced by supplier *B*.

10.1 Inferences About the Difference Between Two Population Means: σ_1 and σ_2 Known

1. μ_1 (μ_2) denote the mean of population 1 (2), we will focus on inferences about the difference between the means: $\mu_1 - \mu_2$.
2. A simple random sample of n_1 (n_2) units from population 1 (2). The two samples, taken separately and independently, are referred to as independent simple random samples.
3. Assume the two population standard deviations, σ_1 and σ_2 , can be assumed known prior to collecting the samples.
4. Question: how to compute a margin of error and develop an interval estimate of the difference between the two population means when σ_1 and σ_2 are known.

Interval Estimation of $\mu_1 - \mu_2$

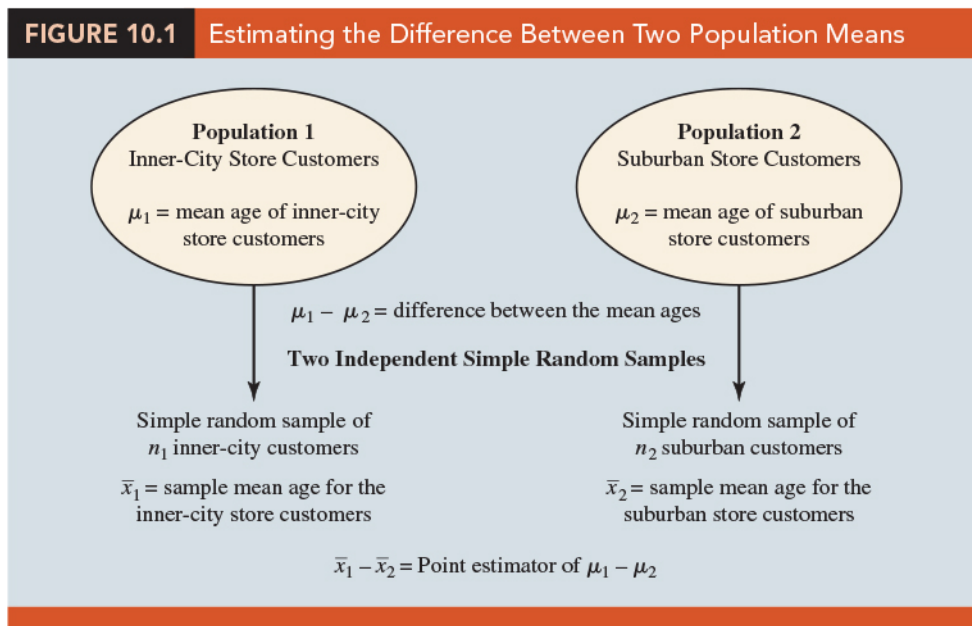
1. **Example** Greystone Department Stores, Inc., operates two stores in Buffalo, New York: One is in the inner city and the other is in a suburban shopping center. The regional manager noticed that products that sell well in one store do not always sell well in the other. The manager believes this situation may be attributable to differences in customer demographics at the two locations. Customers may differ in age, education, income, and so on. (對觀察事物提出問題)
2. Suppose the manager asks us to investigate the difference between the mean ages of the customers who shop at the two stores. (針對問題收集資料)
3. Let us define population 1 as all customers who shop at the inner-city store and population 2 as all customers who shop at the suburban store.
 - (a) μ_1 : mean of population 1 (i.e., the mean age of all customers who shop at the inner-city store)
 - (b) μ_2 : mean of population 2 (i.e., the mean age of all customers who shop at the suburban store)

4. The difference between the two population means is $\mu_1 - \mu_2$. To estimate $\mu_1 - \mu_2$, we will select a simple random sample of n_1 customers from population 1 and a simple random sample of n_2 customers from population 2.
5. We then compute the two sample means.
 - (a) \bar{x}_1 : sample mean age for the simple random sample of n_1 inner-city customers
 - (b) \bar{x}_2 : sample mean age for the simple random sample of n_2 suburban customers
6. The point estimator of the difference between the two population means is the difference between the two sample means.

7. Point Estimator of the Difference Between Two Population Means

$$\hat{\mu}_1 - \hat{\mu}_2 = \bar{x}_1 - \bar{x}_2 \quad (10.1)$$

8. (Figure 10.1) the process used to estimate the difference between two population means based on two independent simple random samples.



9. The point estimator $\bar{x}_1 - \bar{x}_2$ has a standard error that describes the variation in the sampling distribution of the estimator.

10. **Standard Error of $\bar{x}_1 - \bar{x}_2$** With two independent simple random samples, the standard error of $\bar{x}_1 - \bar{x}_2$ is as follows:

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \quad (10.2)$$

(証明如下:) (Hint: $Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X, Y)$).

11. If both populations have a normal distribution, or if the sample sizes are large enough that the central limit theorem enables us to conclude that the sampling distributions of \bar{x}_1 and \bar{x}_2 can be approximated by a normal distribution, the sampling distribution of $\bar{x}_1 - \bar{x}_2$ will have a normal distribution with mean given by $\mu_1 - \mu_2$. (Denoted by $\bar{X}_1 - \bar{X}_2 \sim N(\mu_1 - \mu_2, \sigma_{\bar{x}_1 - \bar{x}_2}^2)$)
12. In general, an interval estimate is given by a point estimate \pm a margin of error. In the case of estimation of the difference between two population means, an interval estimate will take the following form:

$$\underline{(\bar{x}_1 - \bar{x}_2) \pm \text{Margin of error}}$$

13. With the sampling distribution of $\bar{x}_1 - \bar{x}_2$ having a normal distribution, we can write the margin of error as follows:

$$\text{Margin of error} = \underline{z_{\alpha/2} \sigma_{\bar{x}_1 - \bar{x}_2}} = \underline{z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \quad (10.3)$$

14. **Interval Estimate of the Difference Between Two Population Means: σ_1 and σ_2 Known**

$$\underline{(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \quad (10.4)$$

where $1 - \alpha$ is the confidence coefficient.

(公式說明如下:)

 **Question** (p485)

Example Greystone example. Based on data from previous customer demographic studies, the two population standard deviations are known with $\sigma_1 = 9$ years and $\sigma_2 = 10$ years. The data collected from the two independent simple random samples of Greystone customers provided the following results.

	Inner City Store	Suburban Store
Sample Size	$n_1 = 36$	$n_2 = 49$
Sample Mean	$\bar{x}_1 = 40$ years	$\bar{x}_2 = 35$ years

Find the margin of error and the 95% confidence interval estimate of the difference between the two population means.

sol:

Hypothesis Tests About $\mu_1 - \mu_2$

- Let us consider hypothesis tests about the difference between two population means. Using D_0 to denote the hypothesized difference between μ_1 and μ_2 , the three

forms for a hypothesis test are as follows:


Left-tailed test	Right-tailed test	Two-tailed test
$H_0 : \underline{\mu_1 - \mu_2 \geq D_0}$	$H_0 : \underline{\mu_1 - \mu_2 \leq D_0}$	$H_0 : \underline{\mu_1 - \mu_2 = D_0}$
$H_a : \underline{\mu_1 - \mu_2 < D_0}$	$H_a : \underline{\mu_1 - \mu_2 > D_0}$	$H_a : \underline{\mu_1 - \mu_2 \neq D_0}$

- In many applications, $D_0 = 0$. Using the two-tailed test as an example, when $D_0 = 0$ the null hypothesis is $H_0 : \mu_1 - \mu_2 = 0$.
- In this case, the null hypothesis is that μ_1 and μ_2 are equal. Rejection of H_0 leads to the conclusion that $H_a : \mu_1 - \mu_2 \neq 0$ is true; that is, μ_1 and μ_2 are not equal.
- The general steps for conducting hypothesis tests: choose a level of significance, compute the value of the test statistic, and find the p-value to determine whether the null hypothesis should be rejected.
- With two independent simple random samples, we showed that the point estimator $\bar{x}_1 - \bar{x}_2$ has a standard error $\sigma_{\bar{x}_1 - \bar{x}_2}$ given by expression (10.2) and, when the sample sizes are large enough, the distribution of $\bar{x}_1 - \bar{x}_2$ can be described by a normal distribution.

6. Test Statistic for Hypothesis Tests About $\mu_1 - \mu_2$: σ_1 and σ_2 Known

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \quad (10.5)$$

- We demonstrated a two-tailed hypothesis test about the difference between two population means. Lower tail and upper tail tests can also be considered. These tests use the same test statistic as given in equation (10.5). The procedure for computing the p -value and the rejection rules for these one-tailed tests are the same as those for hypothesis tests involving a single population mean and single population proportion.

 Question (p486)

As part of a study to evaluate differences in education quality between two training centers, a standardized examination is given to individuals who are trained at the centers. The difference between the mean examination scores is used to assess quality differences between the centers. The population means for the two centers are as follows. μ_1 is the mean examination score for the population of individuals trained at center A , μ_2 is the mean examination score for the population of individuals trained at center B . We begin with the tentative assumption that no difference exists between the training quality provided at the two centers. The standardized examination given previously in a variety of settings always resulted in an examination score standard deviation near 10 points. Thus, we will use this information to assume that the population standard deviations are known with $\sigma_1 = 10$ and $\sigma_2 = 10$. An $\alpha = 0.05$ level of significance is specified for the study. Independent simple random samples of $n_1 = 30$ individuals from training center A and $n_2 = 40$ individuals from training center B are taken. The respective sample means are $\bar{x}_1 = 82$ and $\bar{x}_2 = 78$. Do these data suggest a significant difference between the population means at the two training centers? State the null and alternative hypotheses for this two-tailed test, compute the test statistic, and state the decision rules based on the p -value approach and the critical value approach and make the decision.

sol:

Practical Advice

1. In most applications of the interval estimation and hypothesis testing procedures presented in this section, random samples with $n_1 \geq 30$ and $n_2 \geq 30$ are adequate.
2. In cases where either or both sample sizes are less than 30, the distributions of the populations become important considerations.
3. In general, with smaller sample sizes, it is more important for the analyst to be satisfied that it is reasonable to assume that the distributions of the two populations are at least approximately normal.

10.2 Inferences About The Difference Between Two Population Means: σ_1 and σ_2 Unknown

1. Extend the discussion of inferences about the difference between two population means to the case when the two population standard deviations, σ_1 and σ_2 , are unknown.
2. In this case, we will use the sample standard deviations, s_1 and s_2 , to estimate the unknown population standard deviations.
3. When we use the sample standard deviations, the interval estimation and hypothesis testing procedures will be based on the t distribution rather than the standard normal distribution.

Interval Estimation of $\mu_1 - \mu_2$

1. Let us develop the margin of error and an interval estimate of the difference between these two population means. (Recall) The interval estimate for the case when the

population standard deviations, σ_1 and σ_2 , are known.

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

2. With σ_1 and σ_2 unknown, we will use the sample standard deviations s_1 and s_2 to estimate σ_1 and σ_2 and replace $z_{\alpha/2}$ with $t_{\alpha/2}$.

3. Interval Estimate of the Difference Between Two Population Means: σ_1 and σ_2 Unknown

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \tag{10.6}$$


where $1 - \alpha$ is the confidence coefficient.

4. In this expression, the use of the t distribution is an approximation, but it provides excellent results and is relatively easy to use. The only difficulty that we encounter in using expression (10.6) is determining the appropriate degrees of freedom for $t_{\alpha/2}$.

5. Statistical software packages compute the appropriate degrees of freedom automatically. The formula used is as follows:

Degrees of Freedom: t Distribution With Two Independent Random Samples

$$df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1-1} \left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2-1} \left(\frac{s_2^2}{n_2}\right)^2} \tag{10.7}$$

 **Question** (p490)

(Clearwater National Bank example) Clearwater National Bank is conducting a study designed to identify differences between checking account practices by customers at two of its branch banks. A simple random sample of 28 checking accounts is selected from the Cherry Grove Branch and an independent simple random sample of 22 checking accounts is selected from the Beechmont Branch. The current checking account balance is recorded for each of the checking accounts. A summary of the account balances follows:

	Cherry Grove	Beechmont
Sample Size	$n_1 = 28$	$n_2 = 22$
Sample Mean	$\bar{x}_1 = \$1025$	$\bar{x}_2 = \$910$
Sample Standard Deviation	$s_1 = \$150$	$s_2 = \$125$

Clearwater National Bank would like to estimate the difference between the mean checking account balance maintained by the population of Cherry Grove customers and the population of Beechmont customers. Compute a 95% confidence interval estimate of the difference between the population mean checking account balances at the two branch banks.

sol:

Hypothesis Tests About $\mu_1 - \mu_2$

- (Recall) Letting D_0 denote the hypothesized difference between μ_1 and μ_2 , the test statistic used for the case where σ_1 and σ_2 are known is as follows.

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$


The test statistic, z , follows the standard normal distribution.

- When σ_1 and σ_2 are unknown, we use s_1 as an estimator of σ_1 and s_2 as an estimator of σ_2 . Substituting these sample standard deviations for σ_1 and σ_2 provides the following test statistic when σ_1 and σ_2 are unknown.

3. Test Statistic for Hypothesis Tests About $\mu_1 - \mu_2$: σ_1 and σ_2 Unknown

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \quad (10.8)$$

The degrees of freedom for t are given by equation (10.7).

 Question (p491)

Consider a new computer software package developed to help systems analysts reduce the time required to design, develop, and implement an information system. To evaluate the benefits of the new software package, a random sample of 24 systems analysts is selected. Each analyst is given specifications for a hypothetical information system. Then 12 of the analysts are instructed to produce the information system by using current technology. The other 12 analysts are trained in the use of the new software package and then instructed to use it to produce the information system. This study involves two populations: a population of systems analysts using the current technology and a population of systems analysts using the new software package. In terms of the time required to complete the information system design project, the population means are as follows. μ_1 is the mean project completion time for systems analysts using the current technology and μ_2 is the mean project completion time for systems analysts using the new software package. The researcher in charge of the new software evaluation project hopes to show that the new software package will provide a shorter mean project completion time. Thus, the researcher is looking for evidence to conclude that μ_2 is less than μ_1 ; in this case, the difference between the two population means, $\mu_1 - \mu_2$, will be greater than zero. Suppose that the 24 analysts complete the study with the results shown in Table 10.1.

TABLE 10.1 Completion Time Data and Summary Statistics for the Software Testing Study		
	Current Technology	New Software
	300	274
	280	220
	344	308
	385	336
	372	198
	360	300
	288	315
	321	258
	376	318
	290	310
	301	332
	283	263
Summary Statistics		
Sample size	$n_1 = 12$	$n_2 = 12$
Sample mean	$\bar{x}_1 = 325$ hours	$\bar{x}_2 = 286$ hours
Sample standard deviation	$s_1 = 40$	$s_2 = 44$

Let the level of significance be $\alpha = 0.05$. State the null and the alternative hypothesis, the test statistic, p -value, the rejection rule, make a decision and conclusion.

sol:

(Software Output)

TABLE 10.2 Output for the Hypothesis Test on the Difference Between the Current and New Software Technology

	Current	New
Mean	325	286
Variance	1600	1936
Observations	12	12
<hr/>		
Hypothesized Mean Difference	0	
Degrees of Freedom	21	
Test Statistic	2.272	
One-Tail p-value	0.017	
One-Tail Critical Value	1.717	

Practical Advice

1. The interval estimation and hypothesis testing procedures presented in this section are robust and can be used with relatively small sample sizes.
2. In most applications, equal or nearly equal sample sizes such that the total sample size $n_1 + n_2 \geq 20$ can be expected to provide very good results even if the populations are not normal.
3. Larger sample sizes are recommended if the distributions of the populations are highly skewed or contain outliers.
4. Smaller sample sizes should only be used if the analyst is satisfied that the distributions of the populations are at least approximately normal.

Notes + Comments

1. How to make inferences about the difference between two population means when σ_1 and σ_2 are equal and unknown ($\sigma_1 = \sigma_2 = \sigma$)?
2. Based on above assumption, the two sample standard deviations are combined to provide the following pooled sample variance:

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

3. The t test statistic becomes

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

and has $n_1 + n_2 - 2$ degrees of freedom. At this point, the computation of the p -value and the interpretation of the sample results are identical to the procedures discussed earlier in this section.

4. A difficulty with this procedure is that the assumption that the two population standard deviations are equal is usually difficult to verify. Unequal population standard deviations are frequently encountered.
5. Using the pooled procedure may not provide satisfactory results, especially if the sample sizes n_1 and n_2 are quite different.
6. The t procedure that we presented in this section does not require the assumption of equal population standard deviations and can be applied whether the population standard deviations are equal or not. It is a more general procedure and is recommended for most applications.

10.3 Inferences About The Difference Between Two Population Means: Matched Samples

1. Example Matched.

- (a) Suppose employees at a manufacturing company can use two different methods to perform a production task. To maximize production output, the company wants to identify the method with the smaller population mean completion time.
- (b) Let μ_1 denote the population mean completion time for production method 1 and μ_2 denote the population mean completion time for production method 2.

- (c) With no preliminary indication of the preferred production method, we begin by tentatively assuming that the two production methods have the same population mean completion time. Thus, the null hypothesis is $H_0 : \mu_1 - \mu_2 = 0$.
- (d) If this hypothesis is rejected, we can conclude that the population mean completion times differ. In this case, the method providing the smaller mean completion time would be recommended.
- (e) The null and alternative hypotheses are written as follows.

$$H_0 : \mu_1 - \mu_2 = 0, \quad H_a : \mu_1 - \mu_2 \neq 0$$

2. In choosing the sampling procedure that will be used to collect production time data and test the hypotheses, we consider two alternative designs. One is based on independent samples and the other is based on matched samples.

- (a) Independent sample design: A simple random sample of workers is selected and each worker in the sample uses method 1. A second independent simple random sample of workers is selected and each worker in this sample uses method 2. The test of the difference between population means is based on the procedures in Section 10.2.
- (b) Matched sample design: One simple random sample of workers is selected. Each worker first uses one method and then uses the other method. The order of the two methods is assigned randomly to the workers, with some workers performing method 1 first and others performing method 2 first. Each worker provides a pair of data values, one value for method 1 and another value for method 2.

3. In the matched sample design the two production methods are tested under similar conditions (i.e., with the same workers); hence this design often leads to a smaller sampling error than the independent sample design. The primary reason is that in a matched sample design, variation between workers is eliminated because the same workers are used for both production methods.

4. Assuming the analysis of a matched sample design is the method used to test the difference between population means for the two production methods. The key

to the analysis of the matched sample design is to realize that we consider only the column of differences, d_i .

5. Therefore, we have six data values (0.6, -0.2, 0.5, 0.3, 0.0, and 0.6) that will be used to analyze the difference between population means of the two production methods.
6. Let μ_d is the mean of the difference in values for the population of workers. With this notation, the null and alternative hypotheses are rewritten as follows.

$$\underline{H_0 : \mu_d = 0, \quad H_a : \mu_d \neq 0}$$


7. Assume the population of differences has a normal distribution. This assumption is necessary so that we may use the t distribution for hypothesis testing and interval estimation procedures. Based on this assumption, the following test statistic has a t distribution with $n-1$ degrees of freedom.

8. **Test Statistic for Hypothesis Tests Involving Matched Samples**

$$\underline{t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}}}$$

where

$$\underline{\bar{d} = \frac{\sum d_i}{n}}, \quad \text{and} \quad \underline{s_d = \sqrt{\frac{\sum (d_i - \bar{d})^2}{n - 1}}} \quad (10.9)$$

 **Question** (p498)

(Table 10.3) (Matched Example). A random sample of six workers is used. The data on completion times for the six workers are given in Table 10.3. Note that each worker provides a pair of data values, one for each production method. Also note that the last column contains the difference in completion times d_i for each worker in the sample. Assume that the population of differences has a normal distribution. Test the hypotheses $H_0 : \mu_d = 0$ and $H_a : \mu_d \neq 0$, using $\alpha = 0.05$. Compute the test statistic, the p -value and draw a conclusion. Compute the 95% confidence interval for the difference between the population means of the two production methods. If H_0 is rejected, we can conclude that the population mean completion times differ.

TABLE 10.3 Task Completion Times for a Matched Sample Design

Worker	Completion Time for Method 1 (minutes)	Completion Time for Method 2 (minutes)	Difference in Completion Times (d_i)
1	6.0	5.4	.6
2	5.0	5.2	-.2
3	7.0	6.5	.5
4	6.2	5.9	.3
5	6.0	6.0	.0
6	6.4	5.8	.6

sol:

Area in Upper Tail	0.20	0.10	0.05	0.025	0.01	0.005
t -Value (5 df)	0.920	1.476	2.015	2.571	3.365	4.032

10.4 Inferences About The Difference Between Two Population Proportions

1. Letting p_1 denote the proportion for population 1 and p_2 denote the proportion for population 2.
2. Consider inferences about the difference between the two population proportions: $p_1 - p_2$.
3. To make an inference about this difference, we will select two independent random samples consisting of n_1 units from population 1 and n_2 units from population 2.

Interval Estimation of $p_1 - p_2$

1. Example Tax Preparation Firm

A tax preparation firm is interested in comparing the quality of work at two of its regional offices. By randomly selecting samples of tax returns prepared at each office and verifying the sample returns' accuracy, the firm will be able to estimate the proportion of erroneous returns prepared at each office. Of particular interest is the difference between these proportions.

- (a) p_1 : proportion of erroneous returns for population 1 (office 1)
- (b) p_2 : proportion of erroneous returns for population 2 (office 2)
- (c) \bar{p}_1 : sample proportion for a simple random sample from population 1
- (d) \bar{p}_2 : sample proportion for a simple random sample from population 2

2. Point Estimator of the Difference Between Two Population Proportions

$$\hat{p}_1 - \hat{p}_2 = \bar{p}_1 - \bar{p}_2 \quad (10.10)$$

3. Thus, the point estimator of the difference between two population proportions is the difference between the sample proportions of two independent simple random samples.
4. As with other point estimators, the point estimator $\bar{p}_1 - \bar{p}_2$ has a sampling distribution that reflects the possible values of $\bar{p}_1 - \bar{p}_2$ if we repeatedly took two independent

random samples. The mean of this sampling distribution is $\underline{p_1 - p_2}$ and the standard error of $\bar{p}_1 - \bar{p}_2$ is:

Standard Error of $\bar{p}_1 - \bar{p}_2$

$$\sigma_{\bar{p}_1 - \bar{p}_2} = \sqrt{\frac{p_1(1 - p_1)}{n_1} + \frac{p_2(1 - p_2)}{n_2}} \quad (10.11)$$


- If the sample sizes are large enough that $\underline{n_1 p_1}$, $\underline{n_1(1-p_1)}$, $\underline{n_2 p_2}$, and $\underline{n_2(1-p_2)}$ are all greater than or equal to $\underline{5}$, the sampling distribution of $\bar{p}_1 - \bar{p}_2$ can be approximated by a normal distribution.
- With the sampling distribution of $\bar{p}_1 - \bar{p}_2$ approximated by a normal distribution, we would like to use $\underline{z_{\alpha/2} \sigma_{\bar{p}_1 - \bar{p}_2}}$ as the margin of error.
- However, $\sigma_{\bar{p}_1 - \bar{p}_2}$ given by equation (10.11) cannot be used directly because the two population proportions, p_1 and p_2 , are unknown. Using the sample proportion \bar{p}_1 to estimate p_1 and the sample proportion \bar{p}_2 to estimate p_2 , the margin of error is:

$$\text{Margin of error} = \underline{z_{\alpha/2} \sqrt{\frac{\bar{p}_1(1 - \bar{p}_1)}{n_1} + \frac{\bar{p}_2(1 - \bar{p}_2)}{n_2}}} \quad (10.12)$$

8. Interval Estimate of the Difference Between Two Population Proportions

$$\underline{(\bar{p}_1 - \bar{p}_2) \pm z_{\alpha/2} \sqrt{\frac{\bar{p}_1(1 - \bar{p}_1)}{n_1} + \frac{\bar{p}_2(1 - \bar{p}_2)}{n_2}}} \quad (10.13)$$

where $1-\alpha$ is the confidence coefficient.

 **Question** (p504)

(Tax Preparation Example) We find that independent simple random samples from the two offices provide the following information.

	Office 1	2
n_i	250	300
Number of returns with errors	35	27

Find a margin of error and interval estimate of the difference between the two population proportions. and 90% confidence interval.

sol:

Hypothesis Tests About $p_1 - p_2$

1. Let us now consider hypothesis tests about no difference between the proportions of two populations. In this case, the three forms for a hypothesis test are as follows:

$$\begin{array}{lll} H_0 : p_1 - p_2 \geq 0, & H_0 : p_1 - p_2 \leq 0, & H_0 : p_1 - p_2 = 0 \\ H_a : p_1 - p_2 < 0 & H_a : p_1 - p_2 > 0 & H_a : p_1 - p_2 \neq 0 \end{array}$$

2. When we assume H_0 is true as an equality, we have $p_1 - p_2 = 0$, which is the same as saying that the population proportions are equal, $p_1 = p_2$.
3. Under the assumption H_0 is true as an equality, the population proportions are equal and $p_1 = p_2 = p$. In this case, $\sigma_{\bar{p}_1 - \bar{p}_2}$ becomes **Standard Error of $\bar{p}_1 - \bar{p}_2$ when $p_1 = p_2 = p$**

$$\sigma_{\bar{p}_1 - \bar{p}_2} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}} = \sqrt{p(1-p) \left(\frac{1}{n_1} + \frac{1}{n_2} \right)} \quad (10.14)$$

4. With p unknown, we pool, or combine, the point estimators from the two samples (\bar{p}_1 and \bar{p}_2) to obtain a single point estimator of p as follows:

Pooled Estimator of p When $p_1 = p_2 = p$


$$\bar{p} = \frac{n_1\bar{p}_1 + n_2\bar{p}_2}{n_1 + n_2} \quad (10.15)$$

This pooled estimator of p is a weighted average of \bar{p}_1 and \bar{p}_2 .

5. Substituting \bar{p} for p in equation (10.14), we obtain an estimate of the standard error of $\bar{p}_1 - \bar{p}_2$. This estimate of the standard error is used in the test statistic.
6. The general form of the test statistic for hypothesis tests about the difference between two population proportions is the point estimator divided by the estimate of $\sigma_{\bar{p}_1 - \bar{p}_2}$.
7. **Test Statistic for Hypothesis Tests About $p_1 - p_2$**

$$z = \frac{\bar{p}_1 - \bar{p}_2}{\sqrt{\bar{p}(1 - \bar{p}) \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}} \quad (10.16)$$

This test statistic applies to large sample situations where n_1p_1 , $n_1(1-p_1)$, n_2p_2 , and $n_2(1-p_2)$ are all greater than or equal to 5.

 **Question** (p506)

(Tax Preparation Firm Example) Assume that the firm wants to use a hypothesis test to determine whether the error proportions differ between the two offices. A two-tailed test is required. Use $\alpha = 0.10$ as the level of significance. State the null and alternative hypotheses. Compute the test statistic, and the p -value for this two-tailed test. State the decision rule and draw a conclusion.

sol:

☺ **EXERCISES**

10.1 : 1, 2, 4, 6

10.2 : 9, 10, 13, 14, 15

10.3 : 19, 23, 24

10.4 : 28, 29, 31, 34

SUP : 38, 39, 44

“少花點時間去取悅別人，多花些時間來經營自己。”

“Spend a little more time trying to make something of yourself and a little less time trying to impress people.”

— 早餐俱樂部 (*Breakfast Club*, 1985)

統計學 (二)

Anderson's Statistics for Business & Economics (14/E)

Chapter 11: Inferences About Population Variances

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Overview

1. Examine methods of statistical inference involving population variances.
2. **Example** **Production Process of Filling Containers**
 - (a) Consider the production process of filling containers with a liquid detergent product. The filling mechanism for the process is adjusted so that the mean filling weight is 450 grams per container.
 - (b) Although a mean of 450 grams is desired, the variance of the filling weights is also critical. That is, even with the filling mechanism properly adjusted for the mean of 450 grams, we cannot expect every container to have exactly 450 grams.
 - (c) By selecting a sample of containers, we can compute a sample variance for the number of grams placed in a container. This value will serve as an estimate of the variance for the population of containers being filled by the production process.

- (d) If the sample variance is modest, the production process will be continued. However, if the sample variance is excessive, overfilling and underfilling may be occurring even though the mean is correct at 450 grams.
- (e) In this case, the filling mechanism will be readjusted in an attempt to reduce the filling variance for the containers.
3. In the first section we consider inferences about the variance of a single population. Subsequently, we will discuss procedures that can be used to make inferences about the variances of two populations.

11.1 Inferences About a Population Variance

1. The sample variance

$$s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1} \quad (11.1)$$

is the point estimator of the population variance σ^2 . In using the sample variance as a basis for making inferences about a population variance, the sampling distribution of the quantity $(n - 1)s^2/\sigma^2$ is helpful.

2. **Sampling Distribution of $(n - 1)s^2/\sigma^2$**

Whenever a simple random sample of size n is selected from a normal population, the sampling distribution of

$$\frac{(n - 1)s^2}{\sigma^2} \quad (11.2)$$

is a chi-square distribution with $n - 1$ degrees of freedom (denoted by $\chi^2(n - 1)$ or χ_{n-1}^2).

3. 補充說明:

- (a) Chi-squared distribution: https://en.wikipedia.org/wiki/Chi-squared_distribution

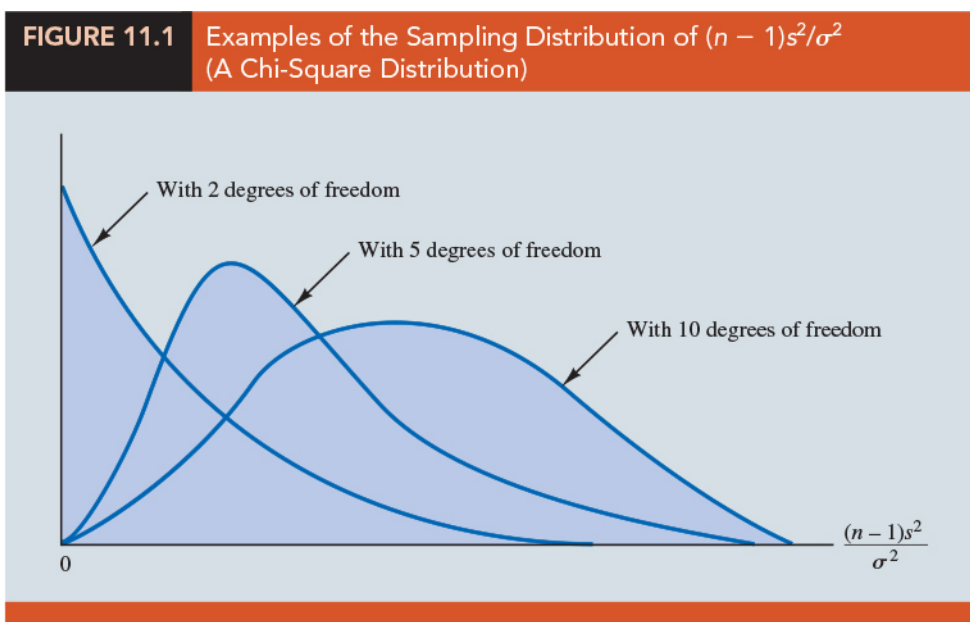
(b) **Theorem:** X_1, X_2, \dots, X_n are observations of a random sample of size n from the normal distribution $N(\mu, \sigma^2)$. \bar{X} is the sample mean and S^2 is the sample variance. Then

i. \bar{X} and S^2 are independent.

ii.
$$\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} \sim \chi^2(n-1)$$

(証明過程) Sampling Distribution of Sample Variance: <https://online.stat.psu.edu/stat414/lesson/26/26.3>

4. (Figure 11.1) shows some possible forms of the sampling distribution of $(n-1)s^2/\sigma^2$. Because the sampling distribution of $(n-1)s^2/\sigma^2$ is known to have a chi-square distribution whenever a simple random sample of size n is selected from a normal population, we can use the chi-square distribution to develop interval estimates and conduct hypothesis tests about a population variance.

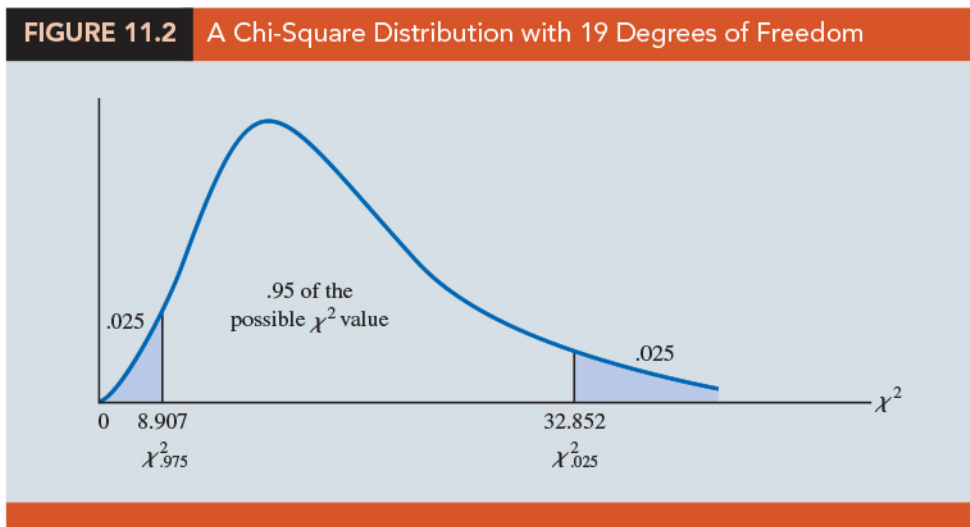


Interval Estimation

- Suppose that we are interested in estimating the population variance for the production filling process. A sample of $n = 20$ containers is taken, and the sample variance for the filling quantities is found to be $s^2 = 2.016$. However, we know we cannot expect the variance of a sample of 20 containers to provide the

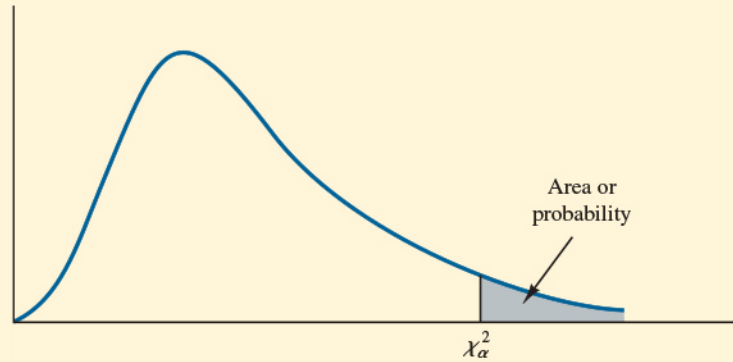
exact value of the variance for the population of containers filled by the production process. Hence, our interest will be in developing an interval estimate for the population variance.

2. (Figure 11.2) We will use the notation χ^2_α to denote the value for the chi-square distribution that provides an area or probability of α to the right of the χ^2_α value.
- (a) the chi-square distribution with 19 degrees of freedom is shown with $\chi^2_{0.025} = 32.852$ indicating that 2.5% of the chi-square values are to the right of 32.852, and
- (b) $\chi^2_{0.975} = 8.907$ indicating that 97.5% of the chi-square values are to the right of 8.907.



3. (Table 11.1) Tables of areas or probabilities are readily available for the chi-square distribution. Table 3 of Appendix B provides a more extensive table of chi-square values.

TABLE 11.1 Selected Values from the Chi-Square Distribution Table*



Degrees of Freedom	Area in Upper Tail							
	.99	.975	.95	.90	.10	.05	.025	.01
1	.000	.001	.004	.016	2.706	3.841	5.024	6.635
2	.020	.051	.103	.211	4.605	5.991	7.378	9.210
3	.115	.216	.352	.584	6.251	7.815	9.348	11.345
4	.297	.484	.711	1.064	7.779	9.488	11.143	13.277
5	.554	.831	1.145	1.610	9.236	11.070	12.832	15.086
6	.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812
7	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475
8	1.647	2.180	2.733	3.490	13.362	15.507	17.535	20.090
9	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666
10	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209
11	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725
12	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217
13	4.107	5.009	5.892	7.041	19.812	22.362	24.736	27.688
14	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141
15	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578
16	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000
17	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409
18	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805
19	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191
20	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566
21	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932
22	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289
23	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638
24	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980
25	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314
26	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642
27	12.878	14.573	16.151	18.114	36.741	40.113	43.195	46.963
28	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278
29	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588
30	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892
40	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691
60	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379
80	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329
100	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807

*Note: A more extensive table is provided as Table 3 of Appendix B.

4. From the graph in Figure 11.2 we see that 0.95, or 95%, of the chi-square values are between $\chi_{0.975}^2$ and $\chi_{0.025}^2$. That is, there is a 0.95 probability of obtaining a χ^2 value such that

$$\chi_{0.975}^2 \leq \chi^2 \leq \chi_{0.025}^2$$

5. We stated in expression (11.2) that $(n-1)s^2/\sigma^2$ follows a chi-square distribution; therefore we can substitute $(n-1)s^2/\sigma^2$ for χ^2 and write

$$\chi_{0.975}^2 \leq \frac{(n-1)s^2}{\sigma^2} \leq \chi_{0.025}^2 \quad (11.3)$$

6. In effect, expression (11.3) provides an interval estimate in that .95, or 95%, of all possible values for $(n-1)s^2/\sigma^2$ will be in the interval $\chi_{0.975}^2$ to $\chi_{0.025}^2$.
7. We now need to do some algebraic manipulations with expression (11.3) to develop an interval estimate for the population variance σ^2 . Working with the leftmost inequality in expression (11.3), we have

$$\chi_{0.975}^2 \leq \frac{(n-1)s^2}{\sigma^2} \Rightarrow \sigma^2 \chi_{0.975}^2 \leq (n-1)s^2 \Rightarrow \sigma^2 \leq \frac{(n-1)s^2}{\chi_{0.975}^2} \quad (11.4)$$

8. Performing similar algebraic manipulations with the rightmost inequality in expression (11.3) gives

$$\frac{(n-1)s^2}{\chi_{0.025}^2} \leq \sigma^2 \quad (11.5)$$

9. The results of expressions (11.4) and (11.5) can be combined to provide a 95% confidence interval estimate for the population variance

$$\frac{(n-1)s^2}{\chi_{0.025}^2} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi_{0.975}^2} \quad (11.6)$$

10. **Example** **Production Process of Filling Containers** Recall that the sample of 20 containers provided a sample variance of $s^2 = 2.016$. With a sample size of 20, we have 19 degrees of freedom and $\chi_{0.975}^2 = 8.907$ and $\chi_{0.025}^2 = 32.852$. Using these values in expression (11.6) provides the following interval estimate for the population variance of filling quantities:

$$\frac{(19)(2.016)}{32.852} \leq \sigma^2 \leq \frac{(19)(2.016)}{8.907} \quad \text{or} \quad 1.166 \leq \sigma^2 \leq 4.300$$

11. Taking the square root of these values provides the following 95% confidence interval for the population standard deviation.

$$\underline{1.080 \leq \sigma \leq 2.074}$$

12. Thus, we illustrated the process of using the chi-square distribution to establish interval estimates of a population variance and a population standard deviation.

13. $(1 - \alpha)\%$ **Confidence Interval Estimate of a Population Variance**

$$\frac{(n-1)s^2}{\chi_{\alpha/2}^2} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi_{(1-\alpha/2)}^2} \quad (11.7)$$

where the χ^2 values are based on a chi-square distribution with $n - 1$ degrees of freedom and where $(1 - \alpha)$ is the confidence coefficient.

Hypothesis Testing

1. Using σ_0^2 to denote the hypothesized value for the population variance, the three forms for a hypothesis test about a population variance are as follows:

$$\begin{array}{lll} H_0 : \sigma^2 \geq \sigma_0^2, & H_0 : \sigma^2 \leq \sigma_0^2, & H_0 : \sigma^2 = \sigma_0^2 \\ H_0 : \sigma^2 < \sigma_0^2, & H_0 : \sigma^2 > \sigma_0^2, & H_0 : \sigma^2 \neq \sigma_0^2 \end{array}$$


2. These three forms are similar to the three forms used to conduct one-tailed and two-tailed hypothesis tests about population means and proportions.
3. The procedure for conducting a hypothesis test about a population variance uses the hypothesized value for the population variance σ_0^2 and the sample variance s^2 to compute the value of a χ^2 test statistic.
4. **Test Statistic for Hypothesis Tests About a Population Variance** Assuming that the population has a normal distribution, the test statistic is:

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} \quad (11.8)$$

where χ^2 has a chi-square distribution with $n - 1$ degrees of freedom.

5. After computing the value of the χ^2 test statistic, either the p-value approach or the critical value approach, may be used to determine whether the null hypothesis can be rejected.
6. Like the t distribution table, the chi-square distribution table does not contain sufficient detail to enable us to determine the p -value exactly. However, we can use the chi-square distribution table to obtain a range for the p -value.
7. (Table 11.2)

TABLE 11.2 Summary of Hypothesis Tests About a Population Variance			
	Lower Tail Test	Upper Tail Test	Two-Tailed Test
Hypotheses	$H_0: \sigma^2 \geq \sigma_0^2$ $H_a: \sigma^2 < \sigma_0^2$	$H_0: \sigma^2 \leq \sigma_0^2$ $H_a: \sigma^2 > \sigma_0^2$	$H_0: \sigma^2 = \sigma_0^2$ $H_a: \sigma^2 \neq \sigma_0^2$
Test Statistic	$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$	$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$	$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$
Rejection Rule: p-value Approach	Reject H_0 if $p\text{-value} \leq \alpha$	Reject H_0 if $p\text{-value} \leq \alpha$	Reject H_0 if $p\text{-value} \leq \alpha$
Rejection Rule: Critical Value Approach	Reject H_0 if $\chi^2 \leq \chi_{(1-\alpha)}^2$	Reject H_0 if $\chi^2 \geq \chi_{\alpha}^2$	Reject H_0 if $\chi^2 \leq \chi_{(1-\alpha/2)}^2$ or if $\chi^2 \geq \chi_{\alpha/2}^2$

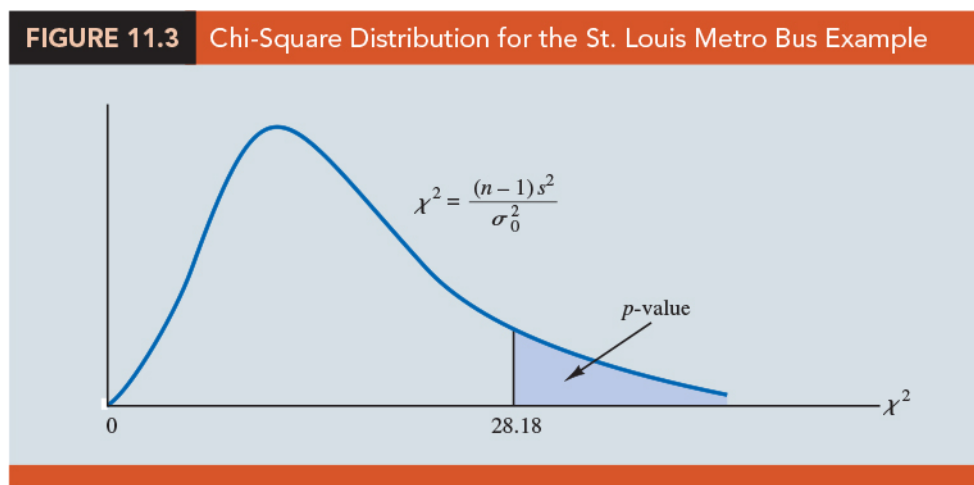
 Question (p532)

(The St. Louis Metro Bus Example) The St. Louis Metro Bus Company wants to promote an image of reliability by encouraging its drivers to maintain consistent schedules. As a standard policy, the company would like arrival times at bus stops to have low variability. In terms of the variance of arrival times, the company standard specifies an arrival time variance of 4 or less when arrival times are measured in minutes. The following hypothesis test is formulated to help the company determine whether the arrival time population variance is excessive.

$$H_0 : \sigma^2 \leq 4, \quad H_a : \sigma^2 > 4$$

In tentatively assuming H_0 is true, we are assuming that the population variance of arrival times is within the company guideline. We reject H_0 if the sample evidence indicates that the population variance exceeds the guideline. In this case, follow-up steps should be taken to reduce the population variance. We conduct the hypothesis test using a level of significance of $\alpha = 0.05$. Suppose that a random sample of 24 bus arrivals taken at a downtown intersection provides a sample variance of $s^2 = 4.9$. Assuming that the population distribution of arrival times is approximately normal. Conduct a hypothesis testing and draw a conclusion using p -value approach and the critical value approach, separately.

sol:



Area in Upper Tail	0.10	0.05	0.025	0.01
χ^2 Value (23 df)	32.007	35.172	38.076	41.638

 Question (p533)

Conduct a two-tailed test about a population variance by considering a situation faced by a bureau of motor vehicles. Historically, the variance in test scores for individuals applying for driver's licenses has been $\sigma^2 = 100$. A new examination with new test questions has been developed. Administrators of the bureau of motor vehicles would like the variance in the test scores for the new examination to remain at the historical level. To evaluate the variance in the new examination test scores, the following two-tailed hypothesis test has been proposed.

$$H_0 : \sigma^2 = 100, \quad H_a : \sigma^2 \neq 100$$

Rejection of H_0 will indicate that a change in the variance has occurred and suggest that some questions in the new examination may need revision to make the variance of the new test scores similar to the variance of the old test scores. A sample of 30 applicants for driver's licenses will be given the new version of the examination. We will use a level of significance $\alpha = 0.05$ to conduct the hypothesis test. The sample of 30 examination scores provided a sample variance $s^2 = 162$. Conduct a hypothesis testing and draw a conclusion using p -value approach.

sol:

Area in Upper Tail	0.10	0.05	0.025	0.01
χ^2 Value (29 df)	39.087	42.557	45.722	49.588

11.2 Inferences About Two Population Variances

1. In some statistical applications we may want to compare the variances in product quality resulting from two different production processes, the variances in assembly times for two assembly methods, or the variances in temperatures for two heating devices.
2. In making comparisons about the two population variances, we will be using data collected from two independent random samples, one from population 1 and another from population 2. The two sample variances s_1^2 and s_2^2 will be the basis for making inferences about the two population variances σ_1 and σ_2 .

3. Sampling Distribution of s_1^2/s_2^2 When $\sigma_1^2 = \sigma_2^2$

Whenever independent simple random samples of sizes n_1 and n_2 are selected from two normal populations with equal variances, the sampling distribution of s_1^2/s_2^2 is an F distribution with $n_1 - 1$ degrees of freedom for the numerator and $n_2 - 1$ degrees of freedom for the denominator; s_1^2 is the sample variance for the random sample of n_1 items from population 1, and s_2^2 is the sample variance for the random sample of n_2 items from population 2.

$$\frac{s_1^2}{s_2^2} \sim F(n_1 - 1, n_2 - 1)$$

4. 補充說明: The sampling distribution of ratio of variances is given by Prof. R. A. Fisher in 1924. According to Prof. R. A. Fisher, the ratio of two independent chi-square variates when divided by their respective degrees of freedom follows F -distribution as

$$F = \frac{\chi_{(n_1-1)}^2/(n_1-1)}{\chi_{(n_2-1)}^2/(n_2-1)}, \quad \text{Since } \chi^2 = \frac{(n-1)s^2}{\sigma^2}, \text{ therefore}$$

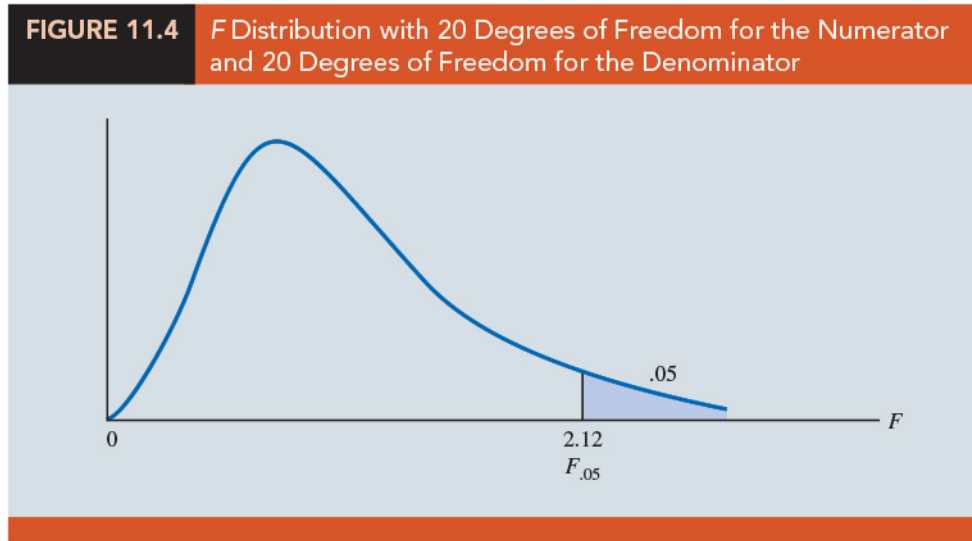
$$F = \frac{((n_1-1)s_1^2/\sigma_1^2)/(n_1-1)}{((n_2-1)s_2^2/\sigma_2^2)/(n_2-1)} = \frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} \sim F_{(n_1-1, n_2-1)}$$

If $\sigma_1^2 = \sigma_2^2$, then

$$F = \frac{s_1^2}{s_2^2} \sim F_{(n_1-1, n_2-1)}$$

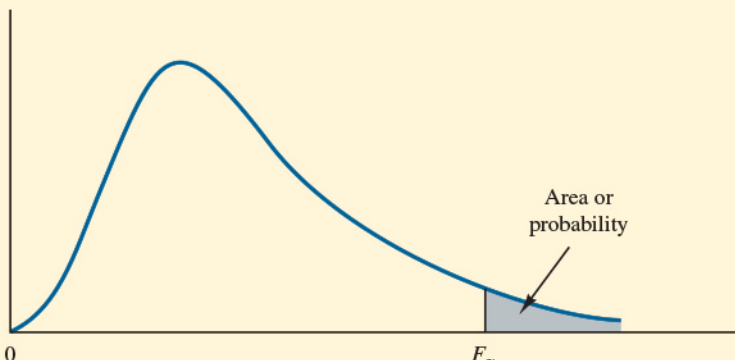
Therefore, the sampling distribution of ratio of sample variances follows F -distribution with $(n_1 - 1, n_2 - 1)$ degrees of freedom.

5. (Figure 11.4) The F distribution is not symmetric and the F values can never be negative. The shape of any particular F distribution depends on its numerator and denominator degrees of freedom.



6. (Table 11.3) We will use F_α to denote the value of F that provides an area or probability of α in the upper tail of the distribution. For example, as noted in Figure 11.4, $F_{0.05}(20, 20) = 2.12$ denotes the upper tail area of 0.05 for an F distribution with 20 degrees of freedom for the numerator and 20 degrees of freedom for the denominator.

TABLE 11.3 Selected Values from the *F* Distribution Table*



Denominator Degrees of Freedom	Area in Upper Tail	Numerator Degrees of Freedom				
		10	15	20	25	30
10	.10	2.32	2.24	2.20	2.17	2.16
	.05	2.98	2.85	2.77	2.73	2.70
	.025	3.72	3.52	3.42	3.35	3.31
	.01	4.85	4.56	4.41	4.31	4.25
15	.10	2.06	1.97	1.92	1.89	1.87
	.05	2.54	2.40	2.33	2.28	2.25
	.025	3.06	2.86	2.76	2.69	2.64
	.01	3.80	3.52	3.37	3.28	3.21
20	.10	1.94	1.84	1.79	1.76	1.74
	.05	2.35	2.20	2.12	2.07	2.04
	.025	2.77	2.57	2.46	2.40	2.35
	.01	3.37	3.09	2.94	2.84	2.78
25	.10	1.87	1.77	1.72	1.68	1.66
	.05	2.24	2.09	2.01	1.96	1.92
	.025	2.61	2.41	2.30	2.23	2.18
	.01	3.13	2.85	2.70	2.60	2.54
30	.10	1.82	1.72	1.67	1.63	1.61
	.05	2.16	2.01	1.93	1.88	1.84
	.025	2.51	2.31	2.20	2.12	2.07
	.01	2.98	2.70	2.55	2.45	2.39

*Note: A more extensive table is provided as Table 4 of Appendix B.

7. Let us show how the *F* distribution can be used to conduct a hypothesis test about the variances of two populations. We begin with a test of the equality of two population variances. The hypotheses are stated as follows.

$$\underline{H_0 : \sigma_1^2 = \sigma_2^2}, \quad \underline{H_a : \sigma_1^2 \neq \sigma_2^2}$$

8. The procedure used to conduct the hypothesis test requires two independent random

samples, one from each population. The two sample variances are then computed. We refer to the population providing the larger sample variance as population 1. Thus, a sample size of n_1 and a sample variance of s_1^2 correspond to population 1, and a sample size of n_2 and a sample variance of s_2^2 correspond to population 2.

9. Test Statistic for Hypothesis Tests About Population Variances With

$$\sigma_1^2 = \sigma_2^2$$

Based on the assumption that both populations have a normal distribution, the ratio of sample variances provides the following F test statistic:

$$F = \frac{s_1^2}{s_2^2} \quad (11.10)$$

Denoting the population with the larger sample variance as population 1, the test statistic has an F distribution with $n_1 - 1$ degrees of freedom for the numerator and $n_2 - 1$ degrees of freedom for the denominator.

10. Because the F test statistic is constructed with the larger sample variance s_1^2 in the numerator, the value of the test statistic will be in the upper tail of the F distribution.
11. (Table 11.4) A summary of hypothesis tests about two population variances.

TABLE 11.4 Summary of Hypothesis Tests About Two Population Variances		
	Upper Tail Test	Two-Tailed Test
Hypotheses	$H_0: \sigma_1^2 \leq \sigma_2^2$ $H_a: \sigma_1^2 > \sigma_2^2$	$H_0: \sigma_1^2 = \sigma_2^2$ $H_a: \sigma_1^2 \neq \sigma_2^2$
		Note: Population 1 has the larger sample variance
Test Statistic	$F = \frac{s_1^2}{s_2^2}$	$F = \frac{s_1^2}{s_2^2}$
Rejection Rule: p-value	Reject H_0 if $p\text{-value} \leq \alpha$	Reject H_0 if $p\text{-value} \leq \alpha$
Rejection Rule: Critical Value Approach	Reject H_0 if $F \geq F_\alpha$	Reject H_0 if $F \geq F_{\alpha/2}$

 Question (p539)

(Dullus County Schools Example) Dullus County Schools is renewing its school bus service contract for the coming year and must select one of two bus companies, the Milbank Company or the Gulf Park Company. We will use the variance of the arrival or pickup/delivery times as a primary measure of the quality of the bus service. Low variance values indicate the more consistent and higher-quality service. If the variances of arrival times associated with the two services are equal, Dullus School administrators will select the company offering the better financial terms. However, if the sample data on bus arrival times for the two companies indicate a significant difference between the variances, the administrators may want to give special consideration to the company with the better or lower variance service. The appropriate hypotheses follow.

$$H_0 : \sigma_1^2 = \sigma_2^2, \quad H_a : \sigma_1^2 \neq \sigma_2^2.$$

If H_0 can be rejected, the conclusion of unequal service quality is appropriate. We will use a level of significance of $\alpha = 0.10$ to conduct the hypothesis test. A sample of 26 arrival times for the Milbank service provides a sample variance of 48 and a sample of 16 arrival times for the Gulf Park service provides a sample variance of 20. Because the Milbank sample provided the larger sample variance, we will denote Milbank as population 1. Use the p -value approach or the critical value approach to obtain the hypothesis testing conclusion.

sol:

Area in Upper Tail	0.10	0.05	0.025	0.01
F Value ($df_1 = 25, df_2 = 15$)	1.89	2.28	2.69	3.28

 Question (p541)

A one-tailed F test about the variances of two populations by considering a public opinion survey. Samples of 31 men and 41 women will be used to study attitudes about current political issues. The researcher conducting the study wants to test to see whether the sample data indicate that women show a greater variation in attitude on political issues than men. In the form of the one-tailed hypothesis test given previously, women will be denoted as population 1 and men will be denoted as population 2. The hypothesis test will be stated as follows.

$$H_0 : \sigma_{women}^2 \leq \sigma_{men}^2, \quad H_a : \sigma_{women}^2 > \sigma_{men}^2.$$

A rejection of H_0 gives the researcher the statistical support necessary to conclude that women show a greater variation in attitude on political issues. The survey results provide a sample variance of $s_1^2 = 120$ for women and a sample variance of $s_2^2 = 80$ for men. Use a level of significance $\alpha = 0.05$ to conduct the hypothesis test.

sol:

☺ **EXERCISES**

11.1 : 2, 3, 5, 9, 10

11.2 : 14, 15, 18, 19

SUP : 26, 29

“永遠不要讓別人的冷漠，影響了你對這世界的熱情。”

“Never allow the indifference of others to affect your passion for this world.”

— 魔女宅急便 (*Kiki's Delivery Service*, 1989)

統計學 (二)

Anderson's Statistics for Business & Economics (14/E)

Chapter 12: Comparing Multiple Proportions, Test of Independence and Goodness of Fit

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Overview

1. Consider cases in which the data are categorical by using a test statistic based on the chi-square (χ^2) distribution.
2. In cases in which data are not naturally categorical, we define categories and consider the observation count in each category. These chi-square tests are versatile and expand hypothesis testing with the following applications.
 - (a) Testing the equality of population proportions for three or more populations. (Chi-Square Test of Homogeneity)
 - (b) Testing the independence of two categorical variables. (Chi-square Test of Independence)
 - (c) Testing whether a probability distribution for a population follows a specific historical or theoretical probability distribution. (Chi-Square Goodness of Fit Test)

12.1 Testing the Equality of Population Proportions for Three or More Populations

1. In this section, we show how the chi-square (χ^2) test statistic can be used to make statistical inferences about the equality of population proportions for three or more populations.
2. Using the notation

p_i = population proportion for population $i, i = 1, 2, 3, \dots, k$.

the hypotheses for the equality of population proportions for $k \geq 3$ populations are as follows:

H_0 : $p_1 = p_2 = \dots = p_k$, H_a : Not all population proportions are equal

- (a) If the sample data and the chi-square test computations indicate H_0 cannot be rejected, we cannot detect a difference among the k population proportions.
 - (b) However, if the sample data and the chi-square test computations indicate H_0 can be rejected, we have the statistical evidence to conclude that not all k population proportions are equal; that is, one or more population proportions differ from the other population proportions.
 - (c) Further analyses can be done to conclude which population proportion or proportions are significantly different from others.
3. **Example** Organizations such as J.D. Power and Associates use the proportion of owners likely to repurchase a particular automobile as an indication of customer loyalty for the automobile. An automobile with a greater proportion of owners likely to repurchase is concluded to have greater customer loyalty.
 - (a) Suppose that in a particular study we want to compare the customer loyalty for three automobiles: Chevrolet Impala, Ford Fusion, and Honda Accord. The current owners of each of the three automobiles form the three populations for the study. The three population proportions of interest are as follows:

p_1 = proportion likely to repurchase an Impala for the population of Chevrolet Impala owners.

p_2 = proportion likely to repurchase a Fusion for the population of Ford Fusion owners.

p_3 = proportion likely to repurchase an Accord for the population of Honda Accord owners.

(b) The hypotheses are stated as follows:

$$H_0 : \underline{p_1 = p_2 = p_3}, \quad H_a : \text{Not all population proportions are equal}$$

(c) To conduct this hypothesis test we begin by taking a sample of owners from each of the three populations. Thus we will have a sample of Chevrolet Impala owners, a sample of Ford Fusion owners, and a sample of Honda Accord owners.

(d) Each sample provides categorical data indicating whether the respondents are likely or not likely to repurchase the automobile.

(e) (Table 12.1) The data for samples of 125 Chevrolet Impala owners, 200 Ford Fusion owners, and 175 Honda Accord owners are summarized in Table 12.1.

		Automobile Owners			
		Chevrolet Impala	Ford Fusion	Honda Accord	Total
Likely to Repurchase	Yes	69	120	123	312
	No	56	80	52	188
	Total	125	200	175	500

(f) This table has two rows for the responses Yes and No and three columns, one corresponding to each of the populations. The observed frequencies are summarized in the six cells of the table corresponding to each combination of the likely to repurchase responses and the three populations.

(g) The data in Table 12.1 are the observed frequencies for each of the six cells that represent the six combinations of the likely to repurchase response and the owner population.

- (h) If we can determine the expected frequencies under the assumption H_0 is true, we can use the chi-square test statistic to determine whether there is a significant difference between the observed and expected frequencies.
- (i) If a significant difference exists between the observed and expected frequencies, the hypothesis H_0 can be rejected and there is evidence that not all the population proportions are equal.
4. Expected frequencies for the six cells of the table are based on the following rationale.
- (a) First, we assume that the null hypothesis of equal population proportions is true.
- (b) Then we note that in the entire sample of 500 owners, a total of 312 owners indicated that they were likely to repurchase their current automobile. Thus, $312/500 = 0.624$ is the overall sample proportion of owners indicating they are likely to repurchase their current automobile.
- (c) If $H_0 : p_1 = p_2 = p_3$ is true, 0.624 would be the best estimate of the proportion responding likely to repurchase for each of the automobile owner populations.
- (d) So if the assumption of H_0 is true, we would expect 0.624 of the 125 Chevrolet Impala owners, or $0.624(125) = 78$ owners to indicate they are likely to repurchase the Impala. Using the 0.624 overall sample proportion, we would expect $0.624(200) = 124.8$ of the 200 Ford Fusion owners and $0.624(175) = 109.2$ of the Honda Accord owners to respond that they are likely to repurchase their respective model of automobile.
5. Let us generalize the approach to computing expected frequencies by letting e_{ij} denote the expected frequency for the cell in row i and column j of the table.
6. Note that 312 is the total number of Yes responses (row 1 total), 125 is the total sample size for Chevrolet Impala owners (column 1 total), and 500 is the total sample size. We can show

$$e_{11} = \left(\frac{\text{Row 1 Total}}{\text{Total Sample Size}} \right) (\text{Column 1 Total}) = \left(\frac{312}{500} \right) (125) = 0.624 \times 125 = 78 .$$

7. Expected Frequencies under the Assumption H_0 is True

$$e_{ij} = \frac{(\text{Row } i \text{ Total})(\text{Column } j \text{ Total})}{\text{Total Sample Size}} \quad (12.1)$$

8. (Table 12.2) Use equation (12.1) to verify the other expected frequencies:

		Automobile Owners			
		Chevrolet Impala	Ford Fusion	Honda Accord	Total
Likely to Repurchase	Yes	78	124.8	109.2	312
	No	47	75.2	65.8	188
	Total	125	200	175	500

9. Chi-Square Test Statistic

$$\chi^2 = \sum_i \sum_j \frac{(f_{ij} - e_{ij})^2}{e_{ij}} \quad (12.2)$$

where

f_{ij} = observed frequency for the cell in row i and column j .

e_{ij} = expected frequency for the cell in row i and column j .

under the assumption H_0 is true.

10. In a chi-square test involving the equality of k population proportions, the above test statistic has a chi-square distribution with $k-1$ degrees of freedom ($\chi^2 \sim \chi_{(k-1)}^2$) provided the expected frequency is 5 or more for each cell.

11. (補充說明) Why the test statistic for the chi-square test of homogeneity has a chi-square distribution? See

(a) The Multinomial Distribution and the Chi-Squared Test for Goodness of Fit:
<https://www.stat.berkeley.edu/~stark/SticiGui/Text/chiSquare.htm>,

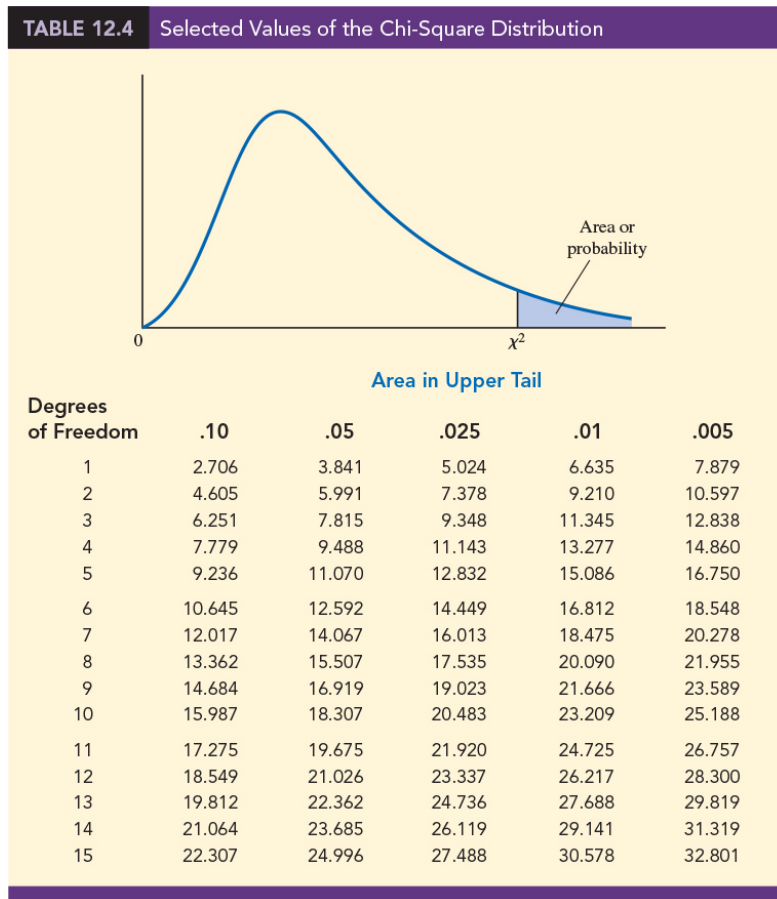
(b) 17.1 - Test For Homogeneity:
<https://online.stat.psu.edu/stat415/lesson/17/17.1>

12. (Table 12.3) Computation of the chi-square test statistic:

TABLE 12.3 Computation of the Chi-Square Test Statistic for the Test of Equal Population Proportions

Likely to Repurchase?	Automobile Owner	Observed Frequency f_{ij}	Expected Frequency e_{ij}	Difference $f_{ij} - e_{ij}$	Squared Difference $(f_{ij} - e_{ij})^2$	Squared Difference Divided by Expected Frequency $(f_{ij} - e_{ij})^2/e_{ij}$
Yes	Impala	69	78.0	-9.0	81.00	1.04
Yes	Fusion	120	124.8	-4.8	23.04	.18
Yes	Accord	123	109.2	13.8	190.44	1.74
No	Impala	56	47.0	9.0	81.00	1.72
No	Fusion	80	75.2	4.8	23.04	.31
No	Accord	52	65.8	-13.8	190.44	2.89
	Total	500	500			$\chi^2 = 7.89$

13. (Table 12.4) In order to understand whether or not the value of the test statistic $\chi^2 = 7.89$ leads us to reject $H_0 : p_1 = p_2 = p_3$, you will need to understand and refer to values of the chi-square distribution:



(a) Since the expected frequencies shown in Table 12.2 are based on the assumption that $H_0 : p_1 = p_2 = p_3$ is true, observed frequencies, f_{ij} , that are in agree-

ment with expected frequencies, e_{ij} , provide small values of $(f_{ij} - e_{ij})^2$ in equation (12.2). If this is the case, the value of the chi-square test statistic will be relatively small and H_0 cannot be rejected.

- (b) On the other hand, if the differences between the observed and expected frequencies are large, values of $(f_{ij} - e_{ij})^2$ and the computed value of the test statistic will be large. In this case, the null hypothesis of equal population proportions can be rejected.
- (c) Thus a chi-square test for equal population proportions will always be an upper tail test with rejection of H_0 occurring when the test statistic is in the upper tail of the chi-square distribution. (Reject H_0 if $\chi^2 \geq \chi_{\alpha, k-1}^2$)
- (d) We can use the upper tail area of the appropriate chi-square distribution and the p-value approach to determine whether the null hypothesis can be rejected.

14. **Example** In the automobile brand loyalty study, the three owner populations indicate that the appropriate chi-square distribution has $k-1 = 3-1 = 2$ degrees of freedom.

chi-square distribution table					
Area in Upper Tail	0.10	0.05	0.025	0.01	0.005
χ^2 Value (2 df)	4.605	5.991	7.378	9.210	10.597

- (a) (The p-value approach) We see the upper tail area at $\chi^2 = 7.89$ is between 0.025 and 0.01. Thus, the corresponding upper tail area or p-value must be between 0.025 and 0.01. (Software: p-value = 0.0193)
- (b) With $p\text{-value} \leq 0.05$, we reject H_0 and conclude that the three population proportions are not all equal and thus there is a difference in brand loyalties among the Chevrolet Impala, Ford Fusion, and Honda Accord owners.
- (c) (The critical value approach) With $\alpha = 0.05$ and 2 degrees of freedom, the critical value for the chi-square test statistic is $\chi_{0.05, 2}^2 = 5.991$. The upper tail rejection region becomes

Reject H_0 if $\chi^2 \geq 5.991$

With $7.89 \geq 5.991$, we reject H_0 .

- (d) Thus, the p -value approach and the critical value approach provide the same hypothesis-testing conclusion.

15. A Chi-Square Test for the Equality of Population Proportions for $k \geq 3$ Populations

- (1) State the null and alternative hypotheses

$$H_0 : p_1 = p_2 = \dots = p_k, \quad H_a : \text{Not all population proportions are equal}$$

- (2) Set the level of significance α . Select a random sample from each of the populations and record the observed frequencies, f_{ij} , in a table with 2 rows and k columns. Assume the null hypothesis is true and compute the expected frequencies, e_{ij} .

- (3) If $e_{ij} \geq 5, \forall i, j$, compute the test statistic:

$$\chi^2 = \sum_i \sum_j \frac{(f_{ij} - e_{ij})^2}{e_{ij}}$$

- (4) Rejection rule (Decision rule):

- i. p -value approach: Reject H_0 if $p\text{-value} \leq \alpha$.
- ii. Critical value approach: Reject H_0 if $\chi^2 \geq \chi_{\alpha, k-1}^2$.

- (5) Make decision.

- (6) Draw conclusion with respect to the problem.

A Multiple Comparison Procedure

1. Since the chi-square test indicated that not all population proportions are equal, it is reasonable for us to proceed by attempting to determine where differences among the population proportions exist.
2. Begin by computing the three sample proportions as follows:

Brand Loyalty	Sample Proportions
Chevrolet Impala	$p_1 = 69/125 = 0.5520$
Ford Fusion	$p_2 = 120/200 = 0.6000$
Honda Accord	$p_3 = 123/175 = 0.7029$

3. For this we will rely on a multiple comparison procedure that can be used to conduct statistical tests between all pairs of population proportions. In the following, we discuss a multiple comparison procedure known as the Marascuilo procedure.
4. We begin by computing the absolute value of the pairwise difference between sample proportions for each pair of populations in the study:

- Chevrolet Impala and Ford Fusion:

$$\underline{|\bar{p}_1 - \bar{p}_2|} = |0.5520 - 0.6000| = 0.0480$$

- Chevrolet Impala and Honda Accord:

$$\underline{|\bar{p}_1 - \bar{p}_3|} = |0.5520 - 0.7029| = 0.1509$$

- Ford Fusion and Honda Accord:

$$\underline{|\bar{p}_2 - \bar{p}_3|} = |0.6000 - 0.7029| = 0.1029$$

5. In a second step, we select a level of significance and compute the corresponding critical value for each pairwise comparison using the following expression.

6. Critical Values for the Marascuilo Pairwise Comparison Procedure for K Population Proportions:

For each pairwise comparison compute a critical value as follows:

$$CV_{ij} = \underline{\sqrt{\chi_\alpha^2} \sqrt{\frac{\bar{p}_i(1 - \bar{p}_i)}{n_i} + \frac{\bar{p}_j(1 - \bar{p}_j)}{n_j}}} \quad (12.3)$$

where

χ_α^2 = chi-square with a level of significance α and $k-1$ degrees of freedom

\bar{p}_i and \bar{p}_j = sample proportions for populations i and j

n_i and n_j = sample sizes for populations i and j

7. Using the chi-square distribution in Table 12.4, $k-1 = 3-1 = 2$ degrees of freedom, and a 0.05 level of significance, we have $\chi_{0.05,2}^2 = 5.991$. Now using the sample proportions $\bar{p}_1 = 0.5520$, $\bar{p}_2 = 0.6000$, and $\bar{p}_3 = 0.7029$, the critical values for the three pairwise comparison tests are as follows:

- (a) Chevrolet Impala and Ford Fusion

$$CV_{12} = \underline{\sqrt{5.991} \sqrt{\frac{0.5520(1 - 0.5520)}{125} + \frac{0.6000(1 - 0.6000)}{200}}} = 0.1380$$

(b) Chevrolet Impala and Honda Accord

$$CV_{13} = \sqrt{5.991} \sqrt{\frac{0.5520(1 - 0.5520)}{125} + \frac{0.7029(1 - 0.7029)}{175}} = 0.1379$$

(c) Ford Fusion and Honda Accord

$$CV_{23} = \sqrt{5.991} \sqrt{\frac{0.6000(1 - 0.6000)}{200} + \frac{0.7029(1 - 0.7029)}{175}} = 0.1198$$

8. If the absolute value of any pairwise sample proportion difference $|\bar{p}_i - \bar{p}_j|$ exceeds its corresponding critical value, CV_{ij} , the pairwise difference is **significant** at the 0.05 level of significance and we can conclude that the two corresponding population proportions are different.
9. (Table 12.5) pairwise comparison procedure:

Pairwise Comparison	$ \bar{p}_i - \bar{p}_j $	CV_{ij}	Significant if $ \bar{p}_i - \bar{p}_j > CV_{ij}$
Chevrolet Impala vs. Ford Fusion	.0480	.1380	Not significant
Chevrolet Impala vs. Honda Accord	.1509	.1379	Significant
Ford Fusion vs. Honda Accord	.1029	.1198	Not significant

10. The conclusion from the pairwise comparison procedure is that the only significant difference in customer loyalty occurs between the Chevrolet Impala and the Honda Accord. Our sample results indicate that the Honda Accord had a greater population proportion of owners who say they are likely to repurchase the Honda Accord. Thus, we can conclude that the Honda Accord ($\bar{p}_3 = 0.7029$) has a greater customer loyalty than the Chevrolet Impala ($\bar{p}_1 = 0.5520$). The results of the study are inconclusive as to the comparative loyalty of the Ford Fusion.
11. While the Ford Fusion did not show significantly different results when compared to the Chevrolet Impala or Honda Accord, a larger sample may have revealed a significant difference between Ford Fusion and the other two automobiles in terms of customer loyalty.

12. It is not uncommon for a multiple comparison procedure to show significance for some pairwise comparisons and yet not show significance for other pairwise comparisons in the study.
13. (補充說明) Why if $|\bar{p}_i - \bar{p}_j| \geq CV_{ij}$, the pairwise difference is significant?
- If $X_1 \sim B(n_1, p_1)$, we have $E(X_1) = n_1 p_1$ and $Var(X_1) = n_1 p_1 (1 - p_1)$.
 - $\frac{X_1}{n_1} = \bar{p}_1 = \hat{p}_1$.
 - $E\left(\frac{X_1}{n_1}\right) =$
 - $Var\left(\frac{X_1}{n_1}\right) =$
 - CLT:
 - Similarly for $X_2 \sim B(n_2, p_2)$.
 - $E(\bar{p}_1 - \bar{p}_2) =$
 - $Var(\bar{p}_1 - \bar{p}_2) =$
 - Under $H_0 : p_1 = p_2$, test statistic: $\bar{p}_1 - \bar{p}_2$

12.2 Test of Independence

1. An important application of a chi-square test involves using sample data to test for the independence of two categorical variables. For this test we take

one sample from a population and record the observations for two categorical variables.

2. We will summarize the data by counting the number of responses for each combination of a category for variable 1 and a category for variable 2.
3. The null hypothesis for this test is that the two categorical variables are independent. Thus, the test is referred to as a test of independence.
4. Example A beer industry association conducts a survey to determine the preferences of beer drinkers for light, regular, and dark beers.
 - (a) A sample of 200 beer drinkers is taken with each person in the sample asked to indicate a preference for one of the three types of beers: light, regular, or dark. At the end of the survey questionnaire, the respondent is asked to provide information on a variety of demographics including gender: male or female.
 - (b) A research question of interest to the association is whether preference for the three types of beer is independent of the gender of the beer drinker.
 - (c) If the two categorical variables, beer preference and gender, are independent, beer preference does not depend on gender and the preference for light, regular, and dark beer can be expected to be the same for male and female beer drinkers.
 - (d) However, if the test conclusion is that the two categorical variables are not independent, we have evidence that beer preference is associated or dependent upon the gender of the beer drinker.
 - (e) As a result, we can expect beer preferences to differ for male and female beer drinkers. In this case, a beer manufacturer could use this information to customize its promotions and advertising for the different target markets of male and female beer drinkers.

5. The hypotheses for this test of independence are as follows:

H_0 : Beer preference is independent of gender

H_a : Beer preference is not independent of gender

6. (Table 12.6) Since an objective of the study is to determine if there is difference between the beer preferences for male and female beer drinkers, we consider gender an

explanatory variable and follow the usual practice of making the explanatory variable the column variable in the data tabulation table. The beer preference is the categorical response variable and is shown as the row variable. The sample results of the 200 beer drinkers in the study are summarized in Table 12.6.

TABLE 12.6 Sample Results for Beer Preferences of Male and Female Beer Drinkers (Observed Frequencies)

		Gender		
		Male	Female	Total
Beer Preference	Light	51	39	90
	Regular	56	21	77
	Dark	25	8	33
	Total	132	68	200

7. For the categorical variable gender, we see 132 of the 200 in the sample were male. This gives us the estimate that $\frac{132}{200} = 0.66$, or 66%, of the beer drinker population is male. Similarly we estimate that $\frac{68}{200} = 0.34$, or 34%, of the beer drinker population is female. Thus male beer drinkers appear to outnumber female beer drinkers approximately 2 to 1.
8. Sample proportions or percentages for the three types of beer are
 - (a) Prefer Light Beer $\frac{90}{200} = 0.450$, or 45.0%
 - (b) Prefer Regular Beer $\frac{77}{200} = 0.385$, or 38.5%
 - (c) Prefer Dark Beer $\frac{33}{200} = 0.165$, or 16.5%
9. Across all beer drinkers in the sample, light beer is preferred most often and dark beer is preferred least often.
10. The computations and formulas used to determine if beer preference and gender are independent are the same as those used for the chi-square test in Section 12.1. Under the assumption that the beer preferences and gender are independent. Thus the expected frequency for row i and column j is given by

$$e_{ij} = \frac{(\text{Row } i \text{ Total})(\text{Column } j \text{ Total})}{\text{Sample Size}} \quad (12.4)$$

11. (Table 12.7) expected frequencies

TABLE 12.7 Expected Frequencies If Beer Preference Is Independent of the Gender of the Beer Drinker

		Gender		
		Male	Female	Total
Beer Preference	Light	59.40	30.60	90
	Regular	50.82	26.18	77
	Dark	21.78	11.22	33
	Total	132	68	200

12. The chi-square test statistic.

$$\chi^2 = \sum_i \sum_j \frac{(f_{ij} - e_{ij})^2}{e_{ij}} \quad (12.5)$$

13. With r rows and c columns in the table, the chi-square distribution will have $\frac{(r-1)(c-1)}{\text{at least 5}}$ degrees of freedom provided the expected frequency is at least 5 for each cell.

14. (Table 12.8)

TABLE 12.8 Computation of the Chi-Square Test Statistic for the Test of Independence Between Beer Preference and Gender

Beer Preference	Gender	Observed Frequency	Expected Frequency	Difference	Squared Difference	Squared Difference Divided by Expected Frequency
		f_{ij}	e_{ij}	$f_{ij} - e_{ij}$	$(f_{ij} - e_{ij})^2$	$(f_{ij} - e_{ij})^2 / e_{ij}$
Light	Male	51	59.40	-8.40	70.56	1.19
Light	Female	39	30.60	8.40	70.56	2.31
Regular	Male	56	50.82	5.18	26.83	.53
Regular	Female	21	26.18	-5.18	26.83	1.02
Dark	Male	25	21.78	3.22	10.37	.48
Dark	Female	8	11.22	-3.22	10.37	.92
Total		200	200			$\chi^2 = 6.45$

15. The upper tail area of the chi-square distribution with 2 degrees of freedom:

Area in Upper Tail	0.10	0.05	0.025	0.01	0.005
χ^2 Value (2 df)	4.605	5.991	7.378	9.210	10.597

16. Thus, we see the upper tail area at $\chi^2 = 6.45$ is between 0.05 and 0.025, and so the corresponding upper tail area or p-value must be between 0.05 and 0.025. With $p\text{-value} \leq .05$, we reject H_0 and conclude that beer preference is not independent of the gender of the beer drinker. (Software: $p\text{-value} = .0398$)
17. With $\alpha = 0.05$ and 2 degrees of freedom, the critical value for the chi-square test statistic is $\chi^2_{0.05,2} = 5.991$. The upper tail rejection region becomes

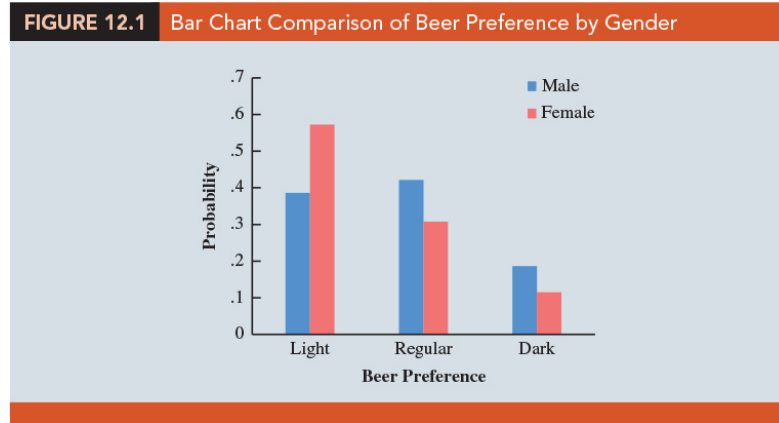
$$\text{Reject } H_0 \text{ if } \chi^2 \geq \chi^2_{0.05,2} = 5.991$$

With $6.45 \geq 5.991$, we reject H_0 .

18. While we now have evidence that beer preference and gender are not independent, we will need to gain additional insight from the data to assess the nature of the association between these two variables. One way to do this is to compute the probability of the beer preference responses for males and females separately.

Beer Preference	Male	Female
Light	$51/132 = 0.3864$, or 38.64%	$39/68 = 0.5735$, or 57.35%
Regular	$56/132 = 0.4242$, or 42.42%	$21/68 = 0.3088$, or 30.88%
Dark	$25/132 = 0.1894$, or 18.94%	$8/68 = .1176$, or 11.76%

19. What observations can you make about the association between beer preference and gender in the sample?
- For female beer drinkers, the highest preference is for light beer at 57.35%.
 - For male beer drinkers, regular beer is most frequently preferred at 42.42%.
 - While female beer drinkers have a higher preference for light beer than males, male beer drinkers have a higher preference for both regular beer and dark beer.
 - (Figure 12.1) Data visualization through bar charts is helpful in gaining insight as to how two categorical variables are associated.



20. Chi-Square Test for Independence of Two Categorical Variables

(1) State the null and alternative hypotheses.

H_0 : The two categorical variables are independent,

H_a : The two categorical variables are not independent

(2) Set a level of significance α . Select a random sample from the population and collect data for both variables for every element in the sample. Record the observed frequencies, f_{ij} , in a table with r rows and c columns. The expected frequencies must all be 5 or more for the chi-square test to be valid. Assume the null hypothesis is true and compute the expected frequencies, e_{ij}

(3) If the expected frequency, e_{ij} , is 5 or more for each cell, compute the test statistic:

$$\chi^2 = \sum_i \sum_j \frac{(f_{ij} - e_{ij})^2}{e_{ij}}$$

(4) Rejection rule:

i. p -value approach: Reject H_0 if $p\text{-value} \leq \alpha$.

ii. Critical value approach: Reject H_0 if $\chi^2 \geq \chi_{\alpha, (r-1)(c-1)}^2$.

(5) Draw decision and conclusion.

21. Finally, if the null hypothesis of independence is rejected, summarizing the probabilities as shown in the above example will help the analyst determine where the association or dependence exists for the two categorical variables.

12.3 Goodness of Fit Test

1. In this section we use a chi-square test (goodness of fit tests) to determine whether a population being sampled has a specific probability distribution.
 - (a) We first consider a population with a historical multinomial probability distribution and use a goodness of fit test to determine if new sample data indicate there has been a change in the population distribution compared to the historical distribution.
 - (b) We then consider a situation where an assumption is made that a population has a normal probability distribution and use a goodness of fit test to determine if sample data indicate that the assumption of a normal probability distribution is or is not appropriate.

Multinomial Probability Distribution

1. With a multinomial probability distribution, each element of a population is assigned to one and only one of three or more categories.
2. Wikipedia: Multinomial distribution:
https://en.wikipedia.org/wiki/Multinomial_distribution
3. Example Consider the market share study being conducted by Scott Marketing Research.
 - (a) Over the past year, market shares for a certain product have stabilized at 30% for company A, 50% for company B, and 20% for company C. Since each customer is classified as buying from one of these companies, we have a multinomial probability distribution with three possible outcomes.
 - (b) The probability for each of the three outcomes is:
 - i. p_A = probability a customer purchases the company A product
 - ii. p_B = probability a customer purchases the company B product
 - iii. p_C = probability a customer purchases the company C product
 - (c) Using the historical market shares, we have multinomial probability distribution with $p_A = 0.30$, $p_B = 0.50$, and $p_C = 0.20$.

- (d) Company *C* plans to introduce a "new and improved" product to replace its current entry in the market. Company *C* has retained Scott Marketing Research to determine whether the new product will alter or change the market shares for the three companies.
 - (e) Specifically, the Scott Marketing Research study will introduce a sample of customers to the new company *C* product and then ask the customers to indicate a preference for the company *A* product, the company *B* product, or the new company *C* product.
4. The hypothesis test to determine if the new company *C* product is likely to change the historical market shares for the three companies.

$$H_0 : p_A = 0.30, p_B = 0.50, \text{ and } p_C = 0.20$$

H_a : The population proportions are not $p_A = 0.30, p_B = 0.50, \text{ and } p_C = 0.20$

5. The null hypothesis is based on the historical multinomial probability distribution for the market shares. If sample results lead to the rejection of H_0 , Scott Marketing Research will have evidence to conclude that the introduction of the new company *C* product will change the market shares.
6. Let us assume that the market research firm has used a consumer panel of 200 customers. Each customer was asked to specify a purchase preference among the three alternatives: company *A*'s product, company *B*'s product, and company *C*'s new product. The 200 responses are summarized:

Observed Frequency		
Company A's Product	Company B's Product	Company C's New Product
48	98	54

7. Perform a goodness of fit test that will determine whether the sample of 200 customer purchase preferences is consistent with the null hypothesis.
8. Like other chi-square tests, the goodness of fit test is based on a comparison of observed frequencies with the expected frequencies under the assumption that the null hypothesis is true.

9. The expected frequency for each category is found by multiplying the sample size of 200 by the hypothesized proportion for the category:

Expected Frequency		
Company A's Product	Company B's Product	Company C's New Product
$200(0.30) = 60$	$200(0.50) = 100$	$200(0.20) = 40$

10. Test Statistic for Goodness of Fit

$$\chi^2 = \sum_{i=1}^k \frac{(f_i - e_i)^2}{e_i} \quad (12.6)$$

where

- (a) $f_i =$ observed frequency for category i
- (b) $e_i =$ expected frequency for category i
- (c) $k =$ the number of categories

Note: The test statistic has a chi-square distribution with $k-1$ degrees of freedom provided that the expected frequencies are 5 or more for all categories.

11. The test for goodness of fit is always a one-tailed test with the rejection occurring in the upper tail of the chi-square distribution:

$$\text{Reject } H_0 \text{ if } \underline{\chi^2 \geq \chi_{\alpha, k-1}^2}$$

12. **Example** (Table 12.9) Let us continue with the Scott Marketing Research example and use the sample data to test the hypothesis that the multinomial population has the market share proportions $p_A = 0.30$, $p_B = 0.50$, and $p_C = 0.20$. We will use an $\alpha = 0.05$ level of significance. We proceed by using the observed and expected frequencies to compute the value of the test statistic.

TABLE 12.9 Computation of the Chi-Square Test Statistic for the Scott Marketing Research Market Share Study

Category	Hypothesized Proportion	Observed Frequency f_i	Expected Frequency e_i	Difference $f_i - e_i$	Squared Difference $(f_i - e_i)^2$	Squared Difference Divided by Expected Frequency $(f_i - e_i)^2/e_i$
Company A	.30	48	60	-12	144	2.40
Company B	.50	98	100	-2	4	.04
Company C	.20	54	40	14	196	4.90
Total		<u>200</u>				<u>$\chi^2 = 7.34$</u>

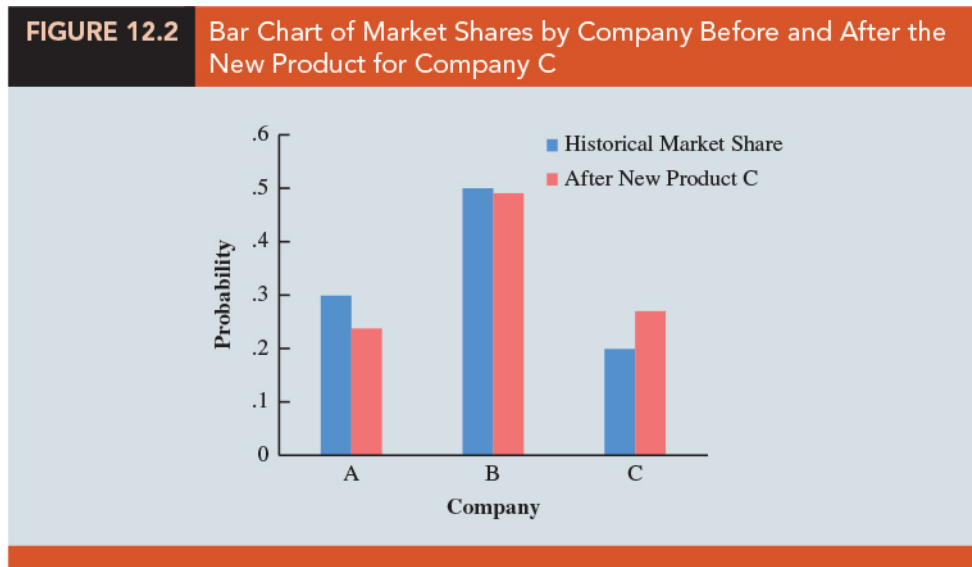
13. With the expected frequencies all 5 or more, the chi-square test statistic is $\chi^2 = 7.34$. We will reject the null hypothesis if the differences between the observed and expected frequencies are large.
14. The test statistic $\chi^2 = 7.34$ is between 5.991 and 7.378. Thus, the corresponding upper tail area or p -value must be between 0.05 and 0.025. With $p\text{-value} \leq 0.05$, we reject H_0 and conclude that the introduction of the new product by company C will alter the historical market shares. (Software: $p\text{-value} = 0.0255$)

Area in Upper Tail	0.10	.05	0.025	0.01	0.005
χ^2 Value (2 df)	4.605	5.991	7.378	9.210	10.597

15. The critical value approach: with $\alpha = 0.05$ and 2 degrees of freedom, the critical value for the test statistic is $\chi_{0.05}^2 = 5.991$. The upper tail rejection rule becomes Reject H_0 if $\chi^2 \geq \chi_{0.05,2}^2 = 5.991$. With $7.34 > 5.991$, we reject H_0 .
16. Now that we have concluded the introduction of a new company C product will alter the market shares for the three companies, we are interested in knowing more about how the market shares are likely to change.
17. Using the historical market shares and the sample data, we summarize the data as follows:

Company	Historical Market Share (%)	Sample Data Market Share (%)
A	30	<u>$48/200 = 0.24$, or 24</u>
B	50	$98/200 = 0.49$, or 49
C	20	$54/200 = 0.27$, or 27

18. (Figure 12.2) This data visualization process shows that the new product will likely increase the market share for company C. Comparisons for the other two companies indicate that company C's gain in market share will hurt company A more than company B.



19. Multinomial Probability Distribution Goodness of Fit Test

- (1) State the null and alternative hypotheses.
 - i. H_0 : The population follows a multinomial probability distribution with specified probabilities for each of the k categories
 - ii. H_a : The population does not follow a multinomial distribution with the specified probabilities for each of the k categories
- (2) Set a level of significance α and select a random sample and record the observed frequencies f_i for each category. Assume the null hypothesis is true and determine the expected frequency e_i in each category by multiplying the category probability by the sample size.
- (3) If the expected frequency e_i is at least 5 for each category, compute the value of the test statistic.

$$\chi^2 = \sum_{i=1}^k \frac{(f_i - e_i)^2}{e_i}$$

- (4) Rejection rule:

- i. p -value approach: Reject H_0 if $p\text{-value} \leq \alpha$
 - ii. Critical value approach: Reject H_0 if $\chi^2 \geq \chi_{\alpha, k-1}^2$.
- (5) Draw decision and conclusion.

Normal Probability Distribution

1. The goodness of fit test for a normal probability distribution is also based on the use of the chi-square distribution.
2. In particular, observed frequencies for several categories of sample data are compared to expected frequencies under the assumption that the population has a normal probability distribution.
3. Because the normal probability distribution is continuous, we must modify the way the categories are defined and how the expected frequencies are computed.
4. Example (Table 12.10) Job applicant test data for Chemline, Inc.

TABLE 12.10 Chemline Employee Aptitude Test Scores for 50 Randomly Chosen Job Applicants					
71	66	61	65	54	93
60	86	70	70	73	73
55	63	56	62	76	54
82	79	76	68	53	58
85	80	56	61	61	64
65	62	90	69	76	79
77	54	64	74	65	65
61	56	63	80	56	71
79	84				

- (a) Chemline hires approximately 400 new employees annually for its four plants located throughout the United States. The personnel director asks whether a normal distribution applies for the population of test scores.
- (b) If such a distribution can be used, the distribution would be helpful in evaluating specific test scores; that is, scores in the upper 20%, lower 40%, and so on, could be identified quickly.

(c) Hence, we want to test the null hypothesis that the population of test scores has a normal distribution.

5. Calculations:

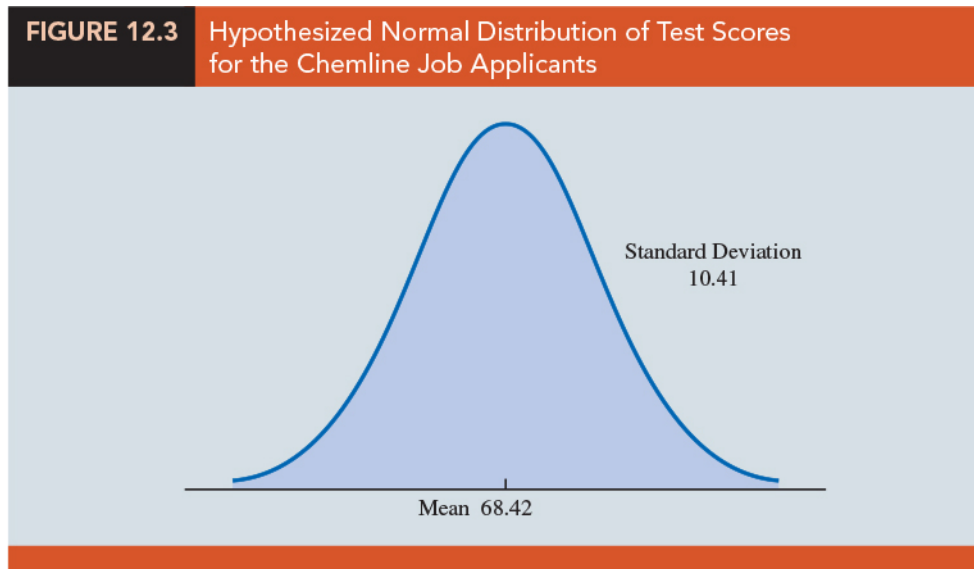
$$\bar{x} = \frac{\sum x_i}{n} = \frac{3421}{50} = 68.42$$

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n - 1}} = \sqrt{\frac{5310.0369}{49}} = 10.41$$

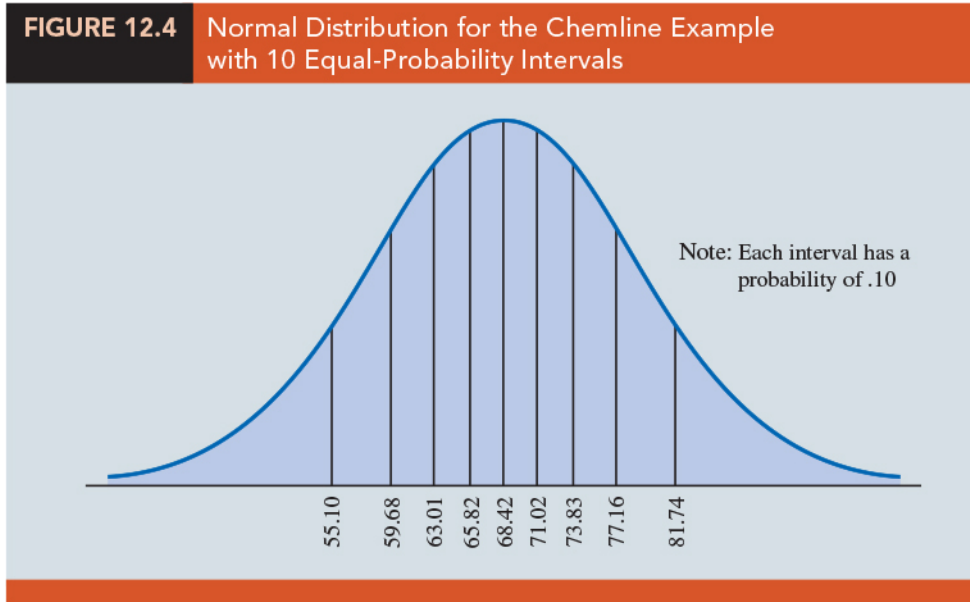
6. Hypotheses about the distribution of the job applicant test scores:

- (a) H_0 : The population of test scores has a normal distribution with mean 68.42 and standard deviation 10.41
- (b) H_a : The population of test scores does not have a normal distribution with mean 68.42 and standard deviation 10.41

7. (Figure 12.3) The hypothesized normal distribution:



8. (Figure 12.4) Define the categories of test scores such that the expected frequencies will be at least five for each category. With a sample size of 50, one way of establishing categories is to divide the normal probability distribution into 10 equal-probability intervals



9. With a sample size of 50, we would expect five outcomes in each interval or category, and the rule of thumb for expected frequencies would be satisfied. Let us look more closely at the procedure for calculating the category boundaries.
- With a continuous probability distribution, establish intervals such that each interval has an expected frequency of five or more.
 - First consider the test score cutting off the lowest 10% of the test scores. From the table for the standard normal distribution we find that the z value for this test score is -1.28. Therefore, the test score of $x = 68.42 - 1.28(10.41) = 55.10$ provides this cutoff value for the lowest 10% of the scores.
 - For the lowest 20%, we find $z = -0.84$, and thus $x = 68.42 - 0.84(10.41) = 59.68$.
 - Working through the normal distribution in that way provides the following test score values:

Percentage	z	Test Score
10%	-1.28	$68.42 - 1.28(10.41) = 55.10$
20%	-.84	$68.42 - .84(10.41) = 59.68$
30%	-.52	$68.42 - .52(10.41) = 63.01$
40%	-.25	$68.42 - .25(10.41) = 65.82$
50%	.00	$68.42 + 0(10.41) = 68.42$
60%	+.25	$68.42 + .25(10.41) = 71.02$
70%	+.52	$68.42 + .52(10.41) = 73.83$
80%	+.84	$68.42 + .84(10.41) = 77.16$
90%	+1.28	$68.42 + 1.28(10.41) = 81.74$

10. (Table 12.11) With the categories or intervals of test scores now defined and with the known expected frequency of five per category, we can return to the sample data of Table 12.10 and determine the observed frequencies for the categories. Doing so provides the results in Table 12.11.

TABLE 12.11 Observed and Expected Frequencies for Chemline Job Applicant Test Scores

Test Score Interval	Observed Frequency f_i	Expected Frequency e_i
Less than 55.10	5	5
55.10 to 59.68	5	5
59.68 to 63.01	9	5
63.01 to 65.82	6	5
65.82 to 68.42	2	5
68.42 to 71.02	5	5
71.02 to 73.83	2	5
73.83 to 77.16	5	5
77.16 to 81.74	5	5
81.74 and over	<u>6</u>	<u>5</u>
Total	50	50

11. (Table 12.12) The value of the test statistic is $\chi^2 = 7.2$.

TABLE 12.12 Computation of the Chi-Square Test Statistic for the Chemline Job Applicant Example

Test Score Interval	Observed Frequency f_i	Expected Frequency e_i	Difference $f_i - e_i$	Squared Difference $(f_i - e_i)^2$	Squared Difference Divided by Expected Frequency $(f_i - e_i)^2 / e_i$
Less than 55.10	5	5	0	0	.0
55.10 to 59.68	5	5	0	0	.0
59.68 to 63.01	9	5	4	16	3.2
63.01 to 65.82	6	5	1	1	.2
65.82 to 68.42	2	5	-3	9	1.8
68.42 to 71.02	5	5	0	0	.0
71.02 to 73.83	2	5	-3	9	1.8
73.83 to 77.16	5	5	0	0	.0
77.16 to 81.74	5	5	0	0	.0
81.74 and over	6	5	1	1	.2
Total	50	50			$\chi^2 = 7.2$

12. Using the rule for computing the number of degrees of freedom for the goodness of fit test, we have $k - p - 1 = 10 - 2 - 1 = 7$ degrees of freedom based on $k = 10$ categories and $p = 2$ parameters (mean and standard deviation) estimated from the sample data.
13. Suppose that we test the null hypothesis that the distribution for the test scores is a normal distribution with a 0.10 level of significance.
14. To test this hypothesis, we need to determine the p -value for the test statistic $\chi^2 = 7.2$ by finding the area in the upper tail of a chi-square distribution with 7 degrees of freedom. (Table 12.4) we find that $\chi^2 = 7.2$ provides an area in the upper tail greater than 0.10. Thus, we know that the p -value is greater than 0.10. (Software: p -value = 0.4084).
15. With $p\text{-value} > 0.10$, the hypothesis that the probability distribution for the Chemline job applicant test scores is a normal probability distribution cannot be rejected.

16. Normal Probability Distribution Goodness of Fit Test

- (1) State the null and alternative hypotheses.

H_0 : The population has a normal probability distribution.

H_a : The population does not have a normal probability distribution.

- (2) Set a level of significance and select a random sample and
 - (a) Compute the sample mean and sample standard deviation.
 - (b) Define k intervals of values so that the expected frequency is at least five for each interval. Using equal probability intervals is a good approach.
 - (c) Record the observed frequency of data values f_i in each interval defined.
- (3) Compute the expected number of occurrences e_i for each interval of values. Multiply the sample size by the probability of a normal random variable being in the interval.
- (4) Compute the value of the test statistic.

$$\chi^2 = \sum_{i=1}^k \frac{(f_i - e_i)^2}{e_i}$$

- (5) Rejection rule:
 - i. p -value approach: Reject H_0 if $p\text{-value} \leq \alpha$
 - ii. Critical value approach: Reject H_0 if $\chi^2 \geq \chi_{\alpha, k-p-1}^2$

where p is the number of parameters of the distribution estimated by the sample.

- (6) Draw decision and conclusion.

☺ **EXERCISES**

12.1 : 1, 2, 3, 7

12.2 : 10, 11, 14, 17

12.3 : 19, 23, 25

SUP : 29, 32, 33, 36

“很多時候我們缺的不是機會，而是決心與勇氣。”

“Often times we lack is not the opportunity, but courage and determination.”

— 心靈補手 (*Good Will Hunting*, 1997)

統計學 (二)

Anderson's Statistics for Business & Economics (14/E)

Chapter 13: Experimental Design and Analysis of Variance

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Overview

1. The statistical studies can be classified as either experimental or observational.
2. In an experimental statistical study, an experiment is conducted to generate the data.
 - (a) An experiment begins with identifying a variable of interest.
 - (b) Then one or more other variables, thought to be related, are identified and controlled, and
 - (c) data are collected about how those variables influence the variable of interest.
3. In an observational study, data are usually obtained through sample surveys and not a controlled experiment.
4. Good design principles are still employed, but the rigorous controls associated with an experimental statistical study are often not possible.

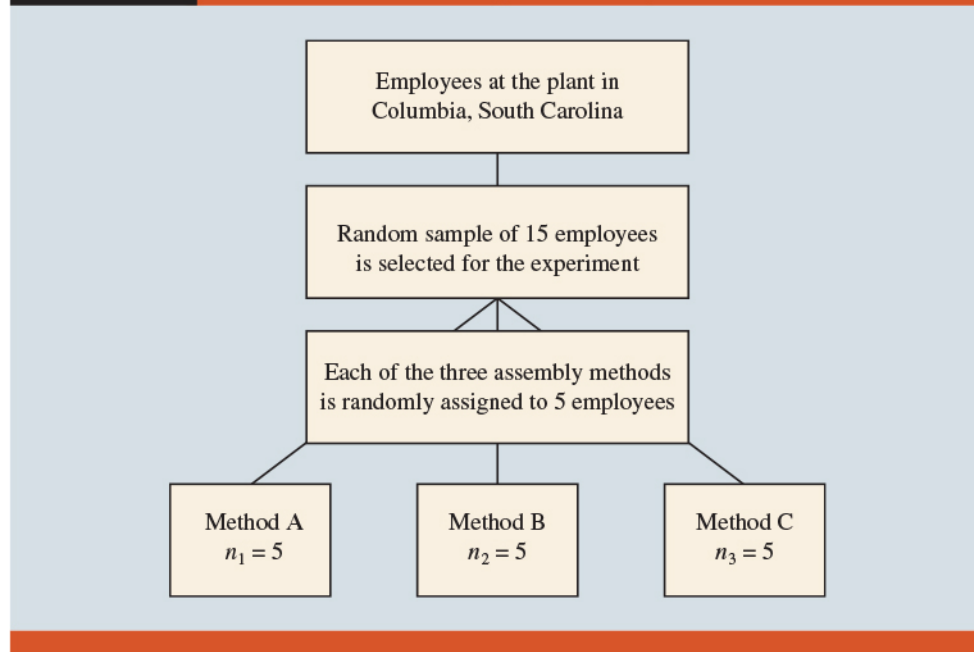
5. **Example** In a study of the relationship between smoking and lung cancer the researcher cannot assign a smoking habit to subjects. The researcher is restricted to simply observing the effects of smoking on people who already smoke and the effects of not smoking on people who do not already smoke.
6. In this chapter we introduce three types of experimental designs: a completely randomized design, a randomized block design*, and a factorial experiment*.
7. Analysis of variance (ANOVA) can analyze the results of regression studies involving both experimental and observational data.

13.1 An Introduction to Experimental Design and Analysis of Variance

1. **Example** Chemitech Inc. developed a new filtration system for municipal water supplies.
 - (a) The industrial engineering group is responsible for determining the best assembly method (method A, method B, and method C) for the new filtration system.
 - (b) Managers at Chemitech want to determine which assembly method can produce the greatest number of filtration systems per week.
 - (c) In the Chemitech experiment, assembly method is the independent variable or factor.
 - (d) Because three assembly methods correspond to this factor, we say that three treatments are associated with this experiment; each treatment corresponds to one of the three assembly methods.
2. **(single-factor experiment)** The Chemitech problem is an example of a single-factor experiment; it involves one categorical factor (method of assembly).

3. More complex experiments may consist of multiple factors; some factors may be categorical and others may be quantitative.
4. (**populations**) The three assembly methods or treatments define the three populations of interest for the Chemitech experiment. One population is all Chemitech employees who use assembly method A, another is those who use method B, and the third is those who use method C.
5. (**objective**) Note that for each population the dependent or response variable is the number of filtration systems assembled per week, and the primary statistical objective of the experiment is to determine whether the mean number of units produced per week is the same for all three populations (methods).
6. (**experimental units**) Suppose a random sample of three employees is selected from all assembly workers at the Chemitech production facility. In experimental design terminology, the three randomly selected workers are the experimental units.
7. (**completely randomized design**) A completely randomized design requires that each of the three assembly methods or treatments be assigned randomly to one of the experimental units or workers.
 - (a) For example, method A might be randomly assigned to the second worker, method B to the first worker, and method C to the third worker.
 - (b) Note that this experiment would result in only one measurement or number of units assembled for each treatment.
8. (**replicates**) To obtain additional data for each assembly method, we must repeat or replicate the basic experimental process.
 - (a) Suppose, for example, we selected 15 workers and then randomly assigned each of the three treatments to 5 of the workers.
 - (b) Because each method of assembly is assigned to 5 workers, we say that five replicates have been obtained.

(Figure 13.1) the completely randomized design for the Chemitech experiment.

FIGURE 13.1 Completely Randomized Design for Evaluating the Chemitech Assembly Method Experiment

Data Collection

1. Once we are satisfied with the experimental design, we proceed by collecting and analyzing the data. In the Chemitech case, the employees would be instructed in how to perform the assembly method assigned to them and then would begin assembling the new filtration systems using that method.
2. (Table 13.1) After this assignment and training, the number of units assembled by each employee during one week is as shown in Table 13.1. The sample means, sample variances, and sample standard deviations for each assembly method are also provided. From these data, method B appears to result in higher production rates than either of the other methods.

TABLE 13.1 Number of Units Produced by 15 Workers

	Method		
	A	B	C
	58	58	48
	64	69	57
	55	71	59
	66	64	47
	67	68	49
Sample mean	62	66	52
Sample variance	27.5	26.5	31.0
Sample standard deviation	5.244	5.148	5.568

- (question) is whether the three sample means observed are different enough for us to conclude that the means of the populations corresponding to the three methods of assembly are different.
- Turn the question to Statistical terms: μ_1, μ_2, μ_3 = mean number of units produced per week using method A, B, C, respectively
- Although we will never know the actual values of μ_1, μ_2 , and μ_3 , we want to use the sample means to test the following hypotheses.

$$H_0 : \mu_1 = \mu_2 = \mu_3, \quad H_a : \text{Not all population means are equal}$$

- The analysis of variance (ANOVA) is the statistical procedure used to determine whether the observed differences in the three sample means are large enough to reject H_0 .

Assumptions for Analysis of Variance

Three assumptions are required to use analysis of variance.

- For each population, the response variable is normally distributed.
Implication: In the Chemitech experiment, the number of units produced per week (response variable) must be normally distributed for each assembly method.
- The variance of the response variable (σ^2), is the same for all of the populations.

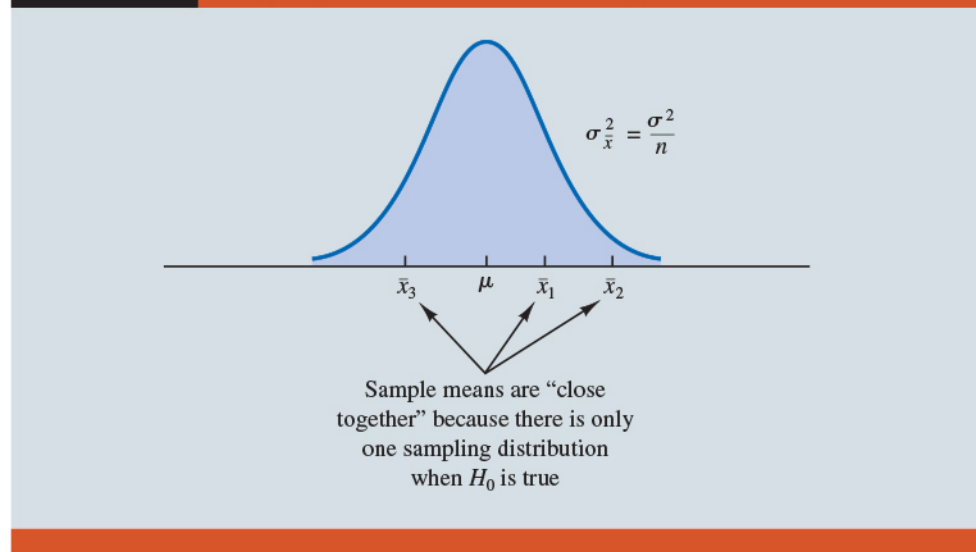
Implication: In the Chemitech experiment, the variance of the number of units produced per week must be the same for each assembly method.

3. The observations must be independent.

Implication: In the Chemitech experiment, the number of units produced per week for each employee must be independent of the number of units produced per week for any other employee.

Analysis of Variance: A Conceptual Overview

1. If the means for the three populations are equal, we would expect the three sample means to be close together.
 - (a) The more the sample means differ, the stronger the evidence we have for the conclusion that the population means differ.
 - (b) If the variability among the sample means is "small," it supports H_0 ; if the variability among the sample means is "large," it supports H_a .
2. If the null hypothesis, $H_0 : \mu_1 = \mu_2 = \mu_3$, is true, we can use the variability among the sample means to develop an estimate of σ^2 .
 - (a) If the assumptions for analysis of variance are satisfied and the null hypothesis is true, each sample will have come from the same normal distribution with mean μ and variance σ^2 .
 - (b) (Chapter 7) the sampling distribution of the sample mean \bar{x} for a simple random sample of size n from a normal population will be normally distributed with mean μ and variance σ^2/n . ($\bar{X}_n \sim N(\mu, \sigma^2/n)$)
 - (c) (Figure 13.2) if H_0 is true, we can think of each of the three sample means, $\bar{x}_1 = 62, \bar{x}_2 = 66$, and $\bar{x}_3 = 52$ from Table 13.1, as values drawn at random from the sampling distribution shown in Figure 13.2.

FIGURE 13.2 Sampling Distribution of \bar{x} Given H_0 Is True

3. When the sample sizes are equal, as in the Chemitech experiment, the best estimate of the mean of the sampling distribution of \bar{x} is the mean or average of the sample means. In the Chemitech experiment, an estimate of the mean of the sampling distribution of \bar{x} is $(62 + 66 + 52)/3 = 60$. We refer to this estimate as the overall sample mean.

4. An estimate of the variance of the sampling distribution of \bar{x} , $\sigma_{\bar{x}}^2$, is provided by the variance of the three sample means.

$$s_{\bar{x}}^2 = \frac{(62 - 60)^2 + (66 - 60)^2 + (52 - 60)^2}{3 - 1} = \frac{104}{2} = 52$$

5. Because $\sigma_{\bar{x}} = \sigma^2/n$, solving for σ^2 gives

$$\sigma^2 = \underline{n\sigma_{\bar{x}}^2}$$

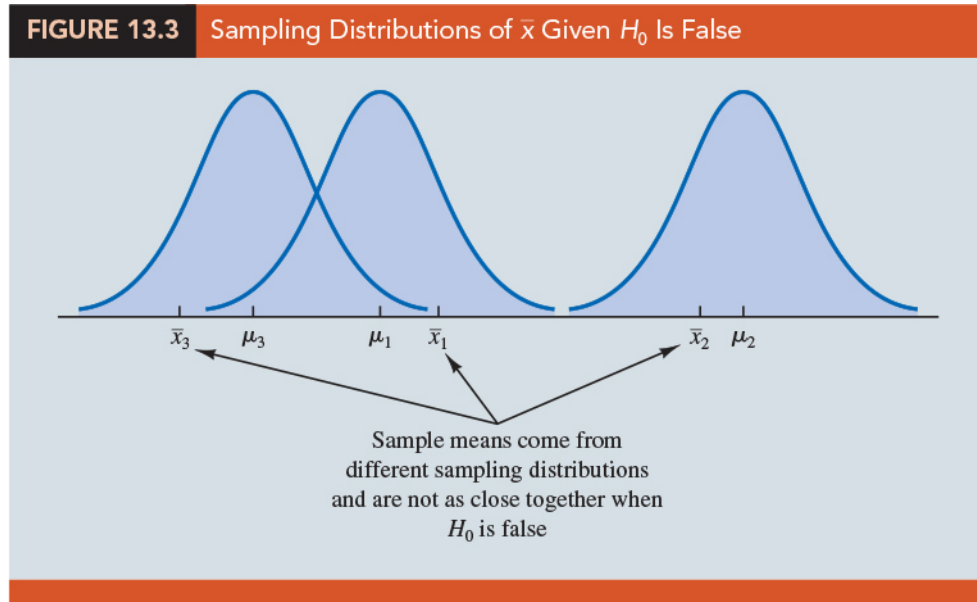
Hence,

$$\text{Estimate of } \sigma^2 = n (\text{Estimate of } \sigma_{\bar{x}}^2) = \underline{ns_{\bar{x}}^2 = 5(52) = 260}.$$

6. The result, $ns_{\bar{x}}^2 = 260$, is referred to as the between-treatments estimate of σ^2 .

7. The between treatments estimate of σ^2 is based on the assumption that H_0 is true. In this case, each sample comes from the same population, and there is only one sampling distribution of \bar{x} .

8. Illustrate what happens when H_0 is false, suppose the population means all differ.
- (a) Note that because the three samples are from normal populations with different means, they will result in three different sampling distributions.
- (b) (Figure 13.3) The sample means are not as close together as they were when H_0 was true. Thus, $s_{\bar{x}}^2$ will be larger, causing the between treatments estimate of σ^2 to be larger.



- (c) In general, when the population means are not equal, the between treatments estimate will overestimate the population variance σ^2 .
- (d) When a simple random sample is selected from each population, each of the sample variances provides an unbiased estimate of σ^2 . Hence, we can combine or pool the individual estimates of σ^2 into one overall estimate.
- (e) The estimate of σ^2 obtained in this way is called the pooled or within-treatments estimate of σ^2 .
- (f) Because each sample variance provides an estimate of σ^2 based only on the variation within each sample, the within treatments estimate of σ^2 is not affected by whether the population means are equal.
- (g) When the sample sizes are equal, the within treatments estimate of σ^2 can be obtained by computing the average of the individual sample variances.

9. **Example** For the Chemitech experiment we obtain

$$\text{Within treatments estimate of } \sigma^2 = \frac{27.5 + 26.5 + 31.0}{3} = \frac{85}{3} = 28.33$$

- (a) The between treatments estimate of σ^2 (260) is much larger than the within treatments estimate of σ^2 (28.33).
- (b) The ratio of these two estimates is $260/28.33 = 9.18$.
10. If the null hypothesis is true,
- (a) The between treatments approach provides a good estimate of σ^2 .
- (b) The two estimates will be similar and their ratio will be close to 1.
11. If the null hypothesis is false,
- (a) The between treatments approach overestimates σ^2
- (b) the between treatments estimate will be larger than the within treatments estimate, and their ratio will be large.
12. In the next section we will show how large this ratio must be to reject H_0 .
13. **Summary:** The logic behind ANOVA is based on the development of two independent estimates of the common population variance σ^2 .
- (a) One estimate of σ^2 is based on the variability among the sample means themselves.
- (b) The other estimate of σ^2 is based on the variability of the data within each sample.
- (c) By comparing these two estimates of σ^2 , we will be able to determine whether the population means are equal.

補充說明:

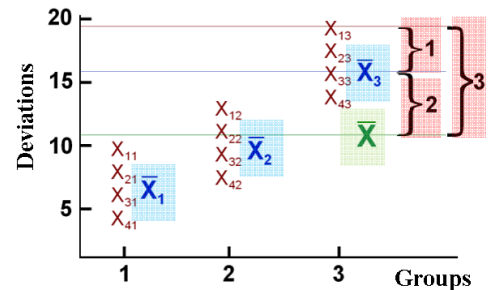
ANOVA Table

Groups					
1	2	...	j	...	k
X_{11}	X_{12}	...	X_{1j}	...	X_{1k}
X_{21}	X_{22}	...	X_{2j}	...	X_{2k}
			...		
X_{i1}	X_{i2}	...	X_{ij}	...	X_{ik}
\vdots			\vdots		
X_{n1}	X_{n2}	...	X_{nj}	...	X_{nk}

$$T_j = \sum_{i=1}^{n_j} X_{ij} \quad \bar{X}_j = \frac{T_j}{n_j}$$

$$T = \sum_{j=1}^k T_j \quad \bar{X} = \frac{T}{N}$$

$$S^2 = \sum_{j=1}^k \sum_{i=1}^{n_j} \frac{(X_{ij} - \bar{X})^2}{N-1}$$



$$(X_{ij} - \bar{X}) = (X_{ij} - \bar{X}_j) + (\bar{X}_j - \bar{X})$$

$$H_0: \mu_1 = \mu_2 = \dots = \mu_k$$

$$X_{ij} = \mu_j + \epsilon_{ij} \quad \begin{matrix} i = 1, \dots, n_j \\ j = 1, \dots, k \end{matrix}$$

$$\epsilon_{ij} \sim N(0, \sigma^2)$$

$$\sum_{j=1}^k \sum_{i=1}^{n_j} (X_{ij} - \bar{X})^2 = \sum_{j=1}^k \sum_{i=1}^{n_j} [(X_{ij} - \bar{X}_j) + (\bar{X}_j - \bar{X})]^2$$

$$\sum_{j=1}^k \sum_{i=1}^{n_j} (X_{ij} - \bar{X})^2 = \sum_{j=1}^k \sum_{i=1}^{n_j} (X_{ij} - \bar{X}_j)^2 + \sum_{j=1}^k \sum_{i=1}^{n_j} (\bar{X}_j - \bar{X})^2$$

ANOVA Table

Source	SS	df	MS	F	p
Between	SS_B	$k-1$	MS_B	MS_B/MS_W	< 0.05
Within	SS_W	$N-k$	MS_W		
Total	SS_T	$N-1$			

$$SS_{Total} = SS_{Within} + SS_{Between}$$

$$F = \frac{MS_{Between}}{MS_{Within}}$$

Reject H_0 , if $F_{obs} > F_{\{\alpha, k-1, N-k\}}$

13.2 Analysis of Variance and the Completely Randomized Design

1. How analysis of variance can be used to test for the equality of k population means for a completely randomized design.

2. The general form of the hypotheses tested is

$$H_0 : \underline{\mu_1 = \mu_2 = \cdots = \mu_k}, \quad H_a : \text{Not all population means are equal}$$

where μ_j is mean of the j th population.

3. We assume that a simple random sample of size $\underline{n_j}$ has been selected from each of the k populations or treatments.

4. For the resulting sample data, let

(a) x_{ij} : value of observation i for treatment j , $i = 1, 2, \dots, n_j$, $j = 1, 2, \dots, k$

(b) n_j : number of observations for treatment j .

(c) \bar{x}_j : sample mean for treatment j ,
$$\bar{x}_j = \frac{\sum_{i=1}^{n_j} x_{ij}}{n_j}.$$

(d) s_j^2 : sample variance for treatment j ,
$$s_j^2 = \frac{\sum_{i=1}^{n_j} (x_{ij} - \bar{x}_j)^2}{n_j - 1}$$

(e) s_j : sample standard deviation for treatment j

5. The overall sample mean, denoted $\underline{\bar{\bar{x}}}$, is the sum of all the observations divided by the total number of observations:

$$\bar{\bar{x}} = \frac{\sum_{j=1}^k \sum_{i=1}^{n_j} x_{ij}}{n_T} \quad (13.3)$$

where $n_T = n_1 + n_2 + \cdots + n_k$ (13.4).

6. If the size of each sample is n , $n_T = kn$; the overall sample mean is just the average of the k sample means.

$$\bar{\bar{x}} = \frac{\sum_{j=1}^k \sum_{i=1}^{n_j} x_{ij}}{kn} = \frac{\sum_{j=1}^k \sum_{i=1}^{n_j} x_{ij}/n}{k} = \frac{\sum_{j=1}^k \bar{x}_j}{k} \quad (13.5)$$

7. The overall sample mean can also be computed as a weighted average of the k sample means.

$$\bar{\bar{x}} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2 + \cdots + n_k \bar{x}_k}{n_T}$$

8. **Example** Each sample in the Chemitech experiment consists of $n = 5$ observations (Table 13.1), we obtained the following result:

$$\bar{x} = \frac{62 + 66 + 52}{3} = 60$$

If the null hypothesis is true ($\mu_1 = \mu_2 = \mu_3 = \mu$), the overall sample mean of 60 is the best estimate of the population mean μ .

Between-Treatments Estimate of Population Variance

1. A between treatments estimate of σ^2 when the sample sizes were equal.
- (a) This estimate of σ^2 is called the mean square due to treatments and is denoted MSTR :

$$MSTR = \frac{\sum_{i=1}^k n_j (\bar{x}_j - \bar{x})^2}{k - 1} \quad (13.6)$$

- (b) The numerator in equation (13.6) is called the sum of squares due to treatments and is denoted SSTR .
- (c) The denominator, $k-1$, represents the degrees of freedom associated with SSTR.
- (d) **Mean Square Due to Treatments**

$$MSTR = \frac{SSTR}{k - 1} \quad (13.7)$$

where

$$SSTR = \sum_{i=1}^k n_j (\bar{x}_j - \bar{x})^2 \quad (13.8)$$

- (e) If H_0 is true, MSTR provides an unbiased estimate of σ^2 . However, if the means of the k populations are not equal, MSTR is not an unbiased estimate of σ^2 ; in fact, in that case, MSTR should overestimate σ^2 .
- (f) If each sample consists of n observations, equation (13.6) can be written as

$$MSTR = \frac{n \sum_{j=1}^k (\bar{x}_j - \bar{x})^2}{k - 1} = n \left[\frac{\sum_{j=1}^k (\bar{x}_j - \bar{x})^2}{k - 1} \right] = \underline{\underline{ns_x^2}}$$

2. **Example** For the Chemitech data in Table 13.1, we obtain the following results:

$$SSTR = \sum_{j=1}^k n_j (\bar{x}_j - \bar{x})^2 = 5(62 - 60)^2 + 5(66 - 60)^2 + 5(52 - 60)^2 = \underline{520}$$

$$MSTR = \frac{SSTR}{k-1} = \frac{520}{2} = \underline{260}$$

Within-Treatments Estimate of Population Variance

1. A within treatments estimate of σ^2 when the sample sizes were equal.

- (a) This estimate of σ^2 is called the mean square due to error and is denoted MSE:

$$MSE = \frac{\sum_{j=1}^k (n_j - 1) s_j^2}{n_T - k} \quad (13.9)$$

- (b) The numerator in equation (13.9) is called the sum of squares due to error and is denoted SSE.

- (c) The denominator of MSE is referred to as the degrees of freedom associated with SSE.

- (d) **Mean Square Due to Error**

$$MSE = \frac{SSE}{n_T - k} \quad (13.10)$$

$$\text{where } SSE = \frac{\sum_{j=1}^k (n_j - 1) s_j^2}{1} \quad (13.11)$$

- (e) Note that MSE is based on the variation within each of the treatments; it is not influenced by whether the null hypothesis is true. Thus, MSE always provides an unbiased estimate of σ^2 .

- (f) If each sample has n observations, $n_T = kn$; thus, $n_T - k = k(n-1)$, and equation (13.9) can be rewritten as

$$MSE = \frac{\sum_{j=1}^k (n-1) s_j^2}{k(n-1)} = \frac{\sum_{j=1}^k s_j^2}{k}$$

- (g) If the sample sizes are the same, MSE is the average of the k sample variances.

- (h) Note that it is the same result we used in Section 13.1 when we introduced the concept of the within-treatments estimate of σ^2 .

2. **Example** For the Chemitech data in Table 13.1 we obtain the following results.

$$SSE = \sum_{j=1}^k (n_j - 1)s_j^2 = (5 - 1)27.5 + (5 - 1)26.5 + (5 - 1)31 = \underline{340}$$

$$MSE = \frac{SSE}{n_T - k} = \frac{340}{15 - 3} = \frac{340}{12} = \underline{28.33}$$

Comparing the Variance Estimates: The F Test

1. If the null hypothesis is true, MSTR and MSE provide two independent, unbiased estimates of σ^2 .
2. (Chapter 11) For normal populations, the sampling distribution of the ratio of two independent estimates of σ^2 follows an F distribution.
3. Hence, if the null hypothesis is true and the ANOVA assumptions are valid, the sampling distribution of $MSTR/MSE$ is an F distribution with numerator degrees of freedom equal to $k-1$ and denominator degrees of freedom equal to $n_T - k$.
 - (a) If the null hypothesis is true, the value of $MSTR/MSE$ should appear to have been selected from this F distribution.
 - (b) If the null hypothesis is false, the value of $MSTR/MSE$ will be inflated because MSTR overestimates σ^2 .
4. Hence, we will reject H_0 if the resulting value of $MSTR/MSE$ appears to be too large to have been selected from an F distribution with $k-1$ numerator degrees of freedom and $n_T - k$ denominator degrees of freedom.

5. Test Statistic for the Equality of K Population Means

$$F = \frac{MSTR}{MSE} \quad (13.12)$$

6. The test statistic follows an F distribution with $k-1$ degrees of freedom in the numerator and $n_T - k$ degrees of freedom in the denominator. ($F \sim F_{k-1, n_T - k}$)
7. **Example** Let us return to the Chemitech experiment and use a level of significance $\alpha = 0.05$ to conduct the hypothesis test.

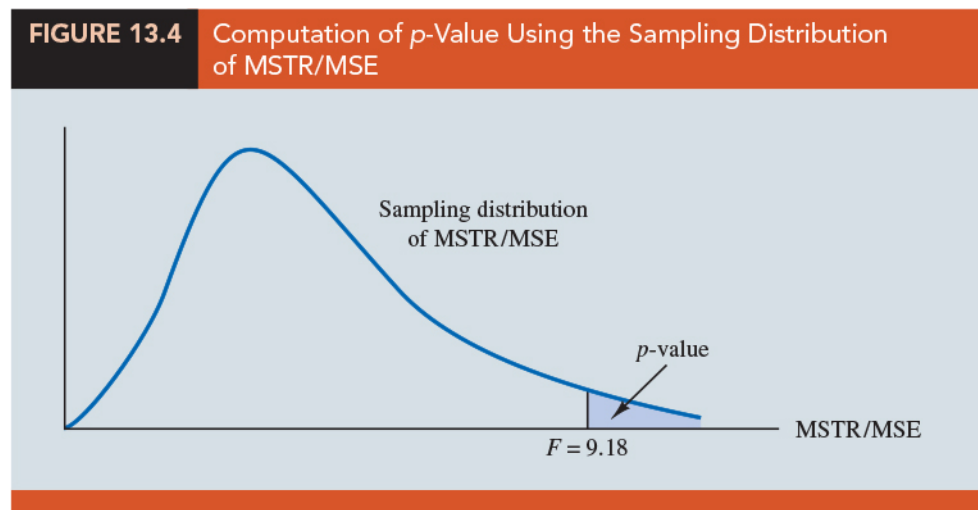
(a) The value of the test statistic is

$$F = \frac{MSTR}{MSE} = \frac{260}{28.33} = 9.18$$

(b) The numerator degrees of freedom is $k-1 = 3-1 = 2$ and the denominator degrees of freedom is $n_T-k = 15-3 = 12$.

(c) Because we will only reject the null hypothesis for large values of the test statistic, the p -value is the upper tail area of the F distribution to the right of the test statistic $F = 9.18$.

(d) (Figure 13.4) shows the sampling distribution of $F = MSTR/MSE$, the value of the test statistic, and the upper tail area that is the p -value for the hypothesis test.



(e) (Table 4 of Appendix B) the upper tail of an F distribution with 2 numerator degrees of freedom and 12 denominator degrees of freedom.

Area in Upper Tail	0.10	0.05	0.025	0.01
F Value ($df_1 = 2, df_2 = 12$)	2.81	3.89	5.10	6.93

(f) (**the p -value approach**) Because $F = 9.18 \geq 6.93$, the area in the upper tail at $F = 9.18$ is less than 0.01. Thus, the p -value is less than 0.01 (Software: p -value = 0.004.)

(g) With $p\text{-value} \leq \alpha = 0.05$, H_0 is rejected.

(h) (**conclusion**) The test provides sufficient evidence to conclude that the means of the three populations are not equal. In other words, analysis of variance

supports the conclusion that the population mean number of units produced per week for the three assembly methods are not equal.

- (i) **(the critical value approach)** With $F = 9.18 \geq F_{(0.05,2,12)} = 3.89$, and conclude that the means of the three populations are not equal.

8. Test for the Equality of K Population Means

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$$

$$H_a : \text{Not all population means are equal}$$

Test Statistic

$$F = \frac{MSTR}{MSE}$$

Rejection Rule

p -value approach : Reject H_0 if $p\text{-value} \leq \alpha$

Critical value approach : Reject H_0 if $F \geq F_{\alpha, k-1, n_T-k}$

ANOVA Table

1. (Table 13.2) The results of the preceding calculations can be displayed conveniently in a table referred to as the analysis of variance or ANOVA table. The general form of the ANOVA table for a completely randomized design is:

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F	p-value
Treatments	SSTR	$k - 1$	$MSTR = \frac{SSTR}{k - 1}$	$\frac{MSTR}{MSE}$	
Error	SSE	$n_T - k$	$MSE = \frac{SSE}{n_T - k}$		
Total	SST	$n_T - 1$			

2. (Table 13.3) JMP/Excel output

TABLE 13.3 Analysis of Variance Table for the Chemitech Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F	p-value
Treatments	520	2	260.00	9.18	.004
Error	340	12	28.33		
Total	860	14			

3. Total sum of squares (SST):

- (a) The sum of squares associated with the source of variation referred to as "Total" is called the total sum of squares (SST).
- (b) $SST = \underline{SSTR + SSE}$, and that the degrees of freedom associated with this total sum of squares is the sum of the degrees of freedom associated with the sum of squares due to treatments and the sum of squares due to error. ($n_T - 1 = (k - 1) + (n_T - k)$.)
- (c) We point out that SST divided by its degrees of freedom $n_T - 1$ is the overall sample variance that would be obtained if we treated the entire set of 15 observations as one data set.
- (d) With the entire data set as one sample, the formula for computing the total sum of squares, SST , is

$$SST = \sum_{j=1}^k \sum_{i=1}^{n_j} (x_{ij} - \bar{x})^2 \quad (13.13)$$

4. ANOVA can be viewed as the process of partitioning the total sum of squares and the degrees of freedom into their corresponding sources: treatments and error.

Computer Results for Analysis of Variance

1. (Figure 13.5) JMP/Excel output for the Chemitech experiment:

FIGURE 13.5 Output for the Chemitech Experiment Analysis of Variance

Source	DF	Adj SS	Adj MS	F-Value	P-Value
Factor	2	520.0	260.00	9.18	.004
Error	12	340.0	28.33		
Total	14	860.0			

Model Summary

S	R-sq	R-sq (adj)
5.32291	60.47%	53.88%

Means

Factor	N	Mean	StDev	95% CI
Method A	5	62.00	5.24	(56.81, 67.19)
Method B	5	66.00	5.15	(60.81, 71.19)
Method C	5	52.00	5.57	(46.81, 57.19)

Pooled StDev = 5.32291

- The square root of MSE provides the best estimate of the population standard deviation σ . This estimate of σ in Figure 13.5 is Pooled StDev; it is equal to 5.323.
- A 95% confidence interval estimate of the population mean for Method A.

$$\bar{x} \pm t_{\alpha/2} \frac{s}{n} \quad (13.15)$$

where s is the estimate of the population standard deviation σ . Because the best estimate of σ is provided by the Pooled StDev, we use a value of 5.323 for σ in expression (13.15).

- The degrees of freedom for the t value is 12, the degrees of freedom associated with the error sum of squares. Hence, with $t_{0.025} = 2.179$ we obtain

$$62 \pm 2.179 \frac{5.323}{\sqrt{5}} = 62 \pm 5.19$$

Thus, the individual 95% confidence interval for Method A goes from $62 - 5.19 = 56.81$ to $62 + 5.19 = 67.19$.

Testing for the Equality of k Population Means: An Observational Study

- ANOVA can also be used to test for the equality of three or more population means using data obtained from an observational study.
- Example** (Table 13.4) National Computer Products, Inc. (NCP) manufactures printers and fax machines at plants located in Atlanta, Dallas, and Seattle. To measure how much employees at these plants know about quality management, a random sample of 6 employees was selected from each plant and the employees selected were given a quality awareness examination. The examination scores for these 18 employees are shown in Table 13.4. Managers want to use these data to test the hypothesis that the mean examination score is the same for all three plants.

	Plant 1 Atlanta	Plant 2 Dallas	Plant 3 Seattle
	85	71	59
	75	75	64
	82	73	62
	76	74	69
	71	69	75
	85	82	67
Sample mean	79	74	66
Sample variance	34	20	32
Sample standard deviation	5.83	4.47	5.66

- Define population 1 as all employees at the Atlanta plant, population 2 as all employees at the Dallas plant, and population 3 as all employees at the Seattle plant. Let μ_j mean examination score for population j , $j = 1, 2, 3$
- Want to use the sample results to test the following hypotheses:

$$H_0 : \mu_1 = \mu_2 = \mu_3$$

$$H_a : \text{Not all population means are equal}$$

- Note that the hypothesis test for the NCP observational study is exactly the same as the hypothesis test for the Chemitech experiment.

6. Even though the same ANOVA methodology is used for the analysis, it is worth noting how the NCP observational statistical study differs from the Chemitech experimental statistical study.
7. The individuals who conducted the NCP study had no control over how the plants were assigned to individual employees. That is, the plants were already in operation and a particular employee worked at one of the three plants. All that NCP could do was to select a random sample of 6 employees from each plant and administer the quality awareness examination.
8. To be classified as an experimental study, NCP would have had to be able to randomly select 18 employees and then assign the plants to each employee in a random fashion.

13.3 Multiple Comparison Procedures

1. When we use analysis of variance to test whether the means of k populations are equal, rejection of the null hypothesis allows us to conclude only that the population means are not all equal.
2. In some cases we will want to go a step further and determine where the differences among means occur.
3. To show how multiple comparison procedures can be used to conduct statistical comparisons between pairs of population means.

Fisher's LSD

1. Suppose that analysis of variance provides statistical evidence to reject the null hypothesis of equal population means. Fisher's least significant difference (LSD) procedure can be used to determine where the differences occur.

2. **Example** (Section 13.1) The Chemitech experiment. Using analysis of variance, we concluded that the mean number of units produced per week are not the same for the three assembly methods. In this case, the followup question is: We believe the assembly methods differ, but where do the differences occur?

3. **Fisher’s LSD Procedure**

$$H_0 : \underline{\mu_i = \mu_j}, \quad H_a : \mu_i \neq \mu_j$$

Test Statistic

$$t = \frac{\bar{x}_i - \bar{x}_j}{\sqrt{MSE \left(\frac{1}{n_i} + \frac{1}{n_j} \right)}} \quad (13.16)$$

Rejection Rule

- *p*-value approach: Reject H_0 if *p*-value $\leq \alpha$.
- Critical value approach: Reject H_0 if $t \leq -t_{\alpha/2}$ or $t \geq t_{\alpha/2}$.

where the value of $t_{\alpha/2}$ is based on a *t* distribution with $n_T - k$ degrees of freedom.

 **Question** (p616)

For the Chemitech experiment, apply Fisher’s LSD Procedure to determine whether there is a significant difference between the means of population 1 (Method A) and population 2 (Method B) at the $\alpha = 0.05$ level of significance.

sol:

Area in Upper Tail	0.20	0.10	0.05	0.025	0.01	0.005
<i>t</i> Value (12 <i>df</i>)	0.873	1.356	1.782	2.179	2.681	3.055

4. Many practitioners find it easier to determine how large the difference between the sample means must be to reject H_0 . In this case the test statistic is $\frac{\bar{x}_i - \bar{x}_j}{\text{LSD}}$, and the test is conducted by the following procedure.

5. **Fisher's LSD Procedure Based on the Test Statistic $\bar{x}_i - \bar{x}_j$**

$$H_0 : \mu_i = \mu_j, \quad H_a : \mu_i \neq \mu_j$$

Test Statistic

$$\frac{\bar{x}_i - \bar{x}_j}{\text{LSD}}$$

Rejection Rule at a Level of Significance α Reject H_0 if $|\bar{x}_i - \bar{x}_j| \geq \text{LSD}$ where

$$\text{LSD} = t_{\alpha/2} \sqrt{MSE \left(\frac{1}{n_i} + \frac{1}{n_j} \right)} \quad (13.17)$$

6. **Confidence Interval Estimate of the Difference Between Two Population Means Using Fisher's LSD Procedure**


$$\frac{(\bar{x}_i - \bar{x}_j) \pm \text{LSD}}{\text{LSD}} \quad (13.18)$$

where

$$\text{LSD} = t_{\alpha/2} \sqrt{MSE \left(\frac{1}{n_i} + \frac{1}{n_j} \right)} \quad (13.19)$$

and $t_{\alpha/2}$ is based on a t distribution with $n_T - k$ degrees of freedom.

7. If the confidence interval in expression (13.18) includes the value zero, we cannot reject the hypothesis that the two population means are equal.
8. However, if the confidence interval does not include the value zero, we conclude that there is a difference between the population means.

 **Question** (p617)

For the Chemitech experiment, apply Fisher's LSD Procedure based on the Test Statistic $\bar{x}_i - \bar{x}_j$ to determine whether there is a significant difference (a) between the means of population 1 (Method A) and population 3 (Method C), (b) between the means of population 2 (Method B) and population 3 (Method C) at the $\alpha = 0.05$ level of significance. Find a 95% confidence interval estimate of the difference between the means of populations 1 and 2 and make a conclusion.

sol:

Type I Error Rates

1. ANOVA gave us statistical evidence to reject or not reject the null hypothesis of equal population means.
2. We showed how Fisher's LSD procedure can be used in such cases to determine where the differences occur. Technically, it is referred to as a protected or restricted LSD test because it is employed only if we first find a significant F value by using analysis of variance.
3. To see why this distinction is important in multiple comparison tests, we need to explain the difference between a comparisonwise Type I error rate and an experimentwise Type I error rate.
4. In the Chemitech experiment we used Fisher's LSD procedure to make three pairwise comparisons.

Test 1	Test 2	Test 3
$H_0 : \mu_1 = \mu_2$	$H_0 : \mu_1 = \mu_3$	$H_0 : \mu_2 = \mu_3$
$H_a : \mu_1 \neq \mu_2$	$H_a : \mu_1 \neq \mu_3$	$H_a : \mu_2 \neq \mu_3$

5. In each case, we used a level of significance of $\alpha = 0.05$.

6. Therefore, for each test, if the null hypothesis is true, the probability that we will make a Type I error is $\alpha = 0.05$; hence, the probability that we will not make a Type I error on each test is $1 - 0.05 = 0.95$.
7. In discussing multiple comparison procedures we refer to this probability of a Type I error ($\alpha = 0.05$) as the comparisonwise Type I error rate; It indicate the level of significance associated with a single pairwise comparison.
8. What is the probability that in making three pairwise comparisons, we will commit a Type I error on at least one of the three tests?
 - (a) To answer this question, note that the probability that we will not make a Type I error on any of the three tests is $(0.95)(0.95)(0.95) = 0.8574$.
 - (b) The probability of making at least one Type I error is $1 - 0.8574 = 0.1426$.
 - (c) Thus, when we use Fisher's LSD procedure to make all three pairwise comparisons, the Type I error rate associated with this approach is not 0.05, but actually 0.1426; we refer to this error rate as the overall or experimentwise Type I error rate.
9. To avoid confusion, we denote the experimentwise Type I error rate as α_{EW} .
10. The experimentwise Type I error rate gets larger for problems with more populations. For example, a problem with five populations has 10 possible pairwise comparisons. If we tested all possible pairwise comparisons by using Fisher's LSD with a comparisonwise error rate of $\alpha = 0.05$, the experimentwise Type I error rate would be $1 - (1 - 0.05)^{10} = 0.40$.
11. In such cases, practitioners look to alternatives that provide better control over the experimentwise error rate.
12. One alternative for controlling the overall experimentwise error rate, referred to as the Bonferroni adjustment, involves using a smaller comparisonwise error rate for each test.
13. For example, if we want to test C pairwise comparisons and want the maximum probability of making a Type I error for the overall experiment to be α_{EW} , we simply use a comparisonwise error rate equal to α_{EW}/C .

14. In the Chemitech experiment, if we want to use Fisher's LSD procedure to test all three pairwise comparisons with a maximum experimentwise error rate of $\alpha_{EW} = 0.05$, we set the comparisonwise error rate to be $\alpha = 0.05/3 = 0.017$.
15. (Recall Chapter 9) For a fixed sample size, any decrease in the probability of making a Type I error will result in an increase in the probability of making a Type II error, which corresponds to accepting the hypothesis that the two population means are equal when in fact they are not equal.
16. As a result, many practitioners are reluctant to perform individual tests with a low comparisonwise Type I error rate because of the increased risk of making a Type II error.
17. Several other procedures, such as Tukey's procedure and Duncan's multiple range test, have been developed to help in such situations. However, there is considerable controversy in the statistical community as to which procedure is "best." The truth is that no one procedure is best for all types of problems.

13.4 Randomized Block Design*

13.5 Factorial Experiment*

☺ EXERCISES

13.2 : 1, 4, 7, 8, 10

13.3 : 13, 15, 18, 19

SUP : 35, 37

“會讓人後悔的從來都不是失敗，而是當機會出現時你沒有全力以赴。”

“Regrets don't come from failure, they come from moments you failed to give your best.”

— 墊底辣妹 (*Flying Colors*, 2015)

統計學 (二)

Anderson's Statistics for Business & Economics (14/E)

Chapter 14: Simple Linear Regression

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Overview

1. Managerial decisions often are based on the relationship between two or more variables.
2. **Examples**
 - (a) After considering the relationship between advertising expenditures and sales, a marketing manager might attempt to predict sales for a given level of advertising expenditures.
 - (b) A public utility might use the relationship between the daily high temperature and the demand for electricity to predict electricity usage on the basis of next month's anticipated daily high temperatures.
3. Regression analysis can be used to develop an equation showing how the variables are related.
4. In regression terminology, the variable being predicted is called the dependent variable (denoted by y). The variable or variables being used to predict the value of the dependent variable are called the independent variables (denoted by x).

5. **Simple linear regression:** the simplest type of regression analysis involving one independent variable and one dependent variable in which the relationship between the variables is approximated by a straight line.
6. Regression analysis involving two or more independent variables is called multiple regression analysis.
7. Multiple regression and cases involving curvilinear relationships are covered in Chapters 15 and 16.

14.1 Simple Linear Regression Model

1. **Example** Armand's Pizza Parlors
 - (a) Armand's Pizza Parlors is a chain of Italian-food restaurants located in a five-state area. Armand's most successful locations are near college campuses.
 - (b) The managers believe that quarterly sales for these restaurants (denoted by y) are related positively to the size of the student population (denoted by x);
 - (c) Restaurants near campuses with a large student population tend to generate more sales than those located near campuses with a small student population.
2. Using regression analysis, we can develop an equation showing how the dependent variable y is related to the independent variable x .

Regression Model and Regression Equation

1. (**population**) In the Armand's Pizza Parlors example, the population consists of all the Armand's restaurants. For every restaurant in the population, there is a value of x (student population) and a corresponding value of y (quarterly sales).

2. (**regression model**) The equation that describes how y is related to x and an error term is called the regression model.

3. Simple Linear Regression Model

$$\underline{y = \beta_0 + \beta_1 x + \epsilon} \quad (14.1)$$

β_0 and β_1 are referred to as the parameters of the model, and ϵ (the Greek letter epsilon) is a random variable referred to as the error term.

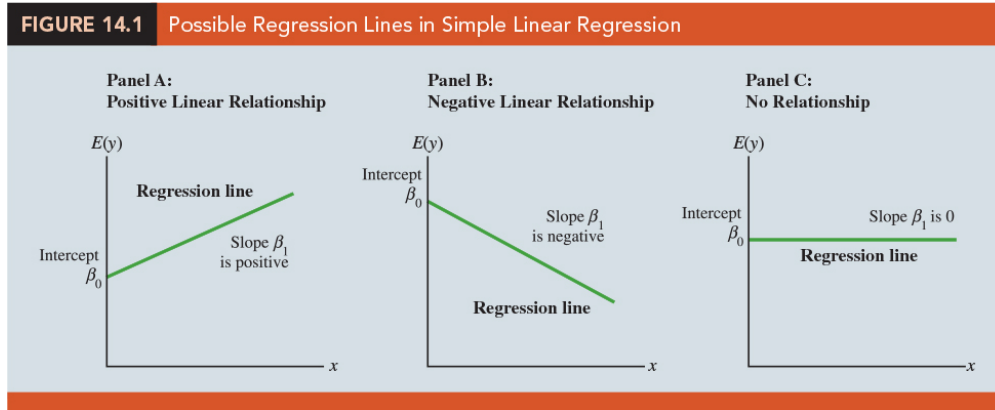
4. The error term accounts for the variability in y that cannot be explained by the linear relationship between x and y .
5. The population of all Armand's restaurants can also be viewed as a collection of subpopulations, one for each distinct value of x .
- (a) For example, one subpopulation consists of all Armand's restaurants located near college campuses with 8000 students; another subpopulation consists of all Armand's restaurants located near college campuses with 9000 students; and so on.
- (b) Each subpopulation has a corresponding distribution of y values. Thus, a distribution of y values is associated with restaurants located near campuses with 8000 students; a distribution of y values is associated with restaurants located near campuses with 9000 students; and so on.
6. (**regression equation**) Each distribution of y values has its own mean or expected value. The equation that describes how the expected value of y , denoted $E(y)$, is related to x is called the regression equation.

7. Simple Linear Regression Equation

$$\underline{E(y) = \beta_0 + \beta_1 x} \quad (14.2)$$

The graph of the simple linear regression equation is a straight line; β_0 is the y -intercept of the regression line, β_1 is the slope, and $E(y)$ is the mean or expected value of y for a given value of x .

8. (Figure 14.1) Possible regression lines



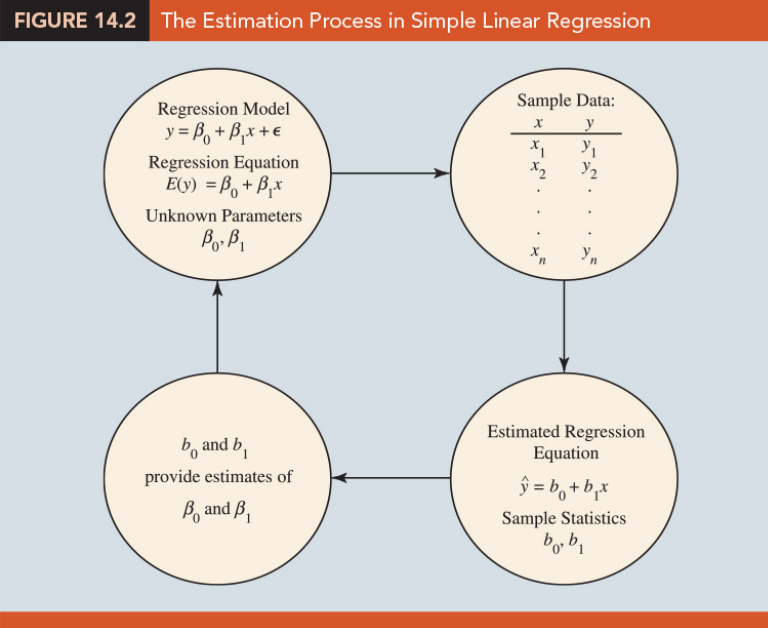
Estimated Regression Equation

1. If the values of the population parameters β_0 and β_1 were known, we could use equation (14.2) to compute the mean value of y for a given value of x .
2. In practice, the parameter values are not known and must be estimated using sample data. Sample statistics (denoted b_0 and b_1) are computed as estimates of the population parameters β_0 and β_1 . Substituting the values of the sample statistics b_0 and b_1 for β_0 and β_1 in the regression equation, we obtain the estimated regression equation.

3. Estimated Simple Linear Regression Equation

$$\hat{y} = b_0 + b_1x \quad (14.3)$$

4. (**the estimated regression line**) The graph of the estimated simple linear regression equation is called the estimated regression line; b_0 is the y -intercept and b_1 is the slope.
5. In general, \hat{y} is the point estimator of $E(y)$, the mean value of y for a given value of x .
6. (Figure 14.2) A summary of the estimation process for simple linear regression.

7. **Example** Armand's Pizza Parlors

- To estimate the mean or expected value of quarterly sales for all restaurants located near campuses with 10,000 students, Armand's would substitute the value of 10,000 for x in equation (14.3).
 - In some cases, however, Armand's may be more interested in predicting sales for one particular restaurant.
 - For example, suppose Armand's would like to predict quarterly sales for the restaurant they are considering building near Talbot College, a school with 10,000 students. As it turns out, the best predictor of y for a given value of x is also provided by \hat{y} .
 - Thus, to predict quarterly sales for the restaurant located near Talbot College, Armand's would also substitute the value of 10,000 for x in equation (14.3).
8. The value of \hat{y} provides both a point estimate of $E(y)$ for a given value of x and a prediction of an individual value of y for a given value of x .

Notes + Comments

- Regression analysis cannot be interpreted as a procedure for establishing a cause-and-effect relationship between variables. It can only indicate how or to what extent variables

are associated with each other.

- Any conclusions about cause and effect must be based upon the judgment of those individuals most knowledgeable about the application.
- The regression equation in simple linear regression is $E(y) = \beta_0 + \beta_1x$. More advanced texts in regression analysis often write the regression equation as

$$\underline{E(y|x) = \beta_0 + \beta_1x}$$

to emphasize that the regression equation provides the mean value of y for a given value of x .

14.2 Least Squares Method

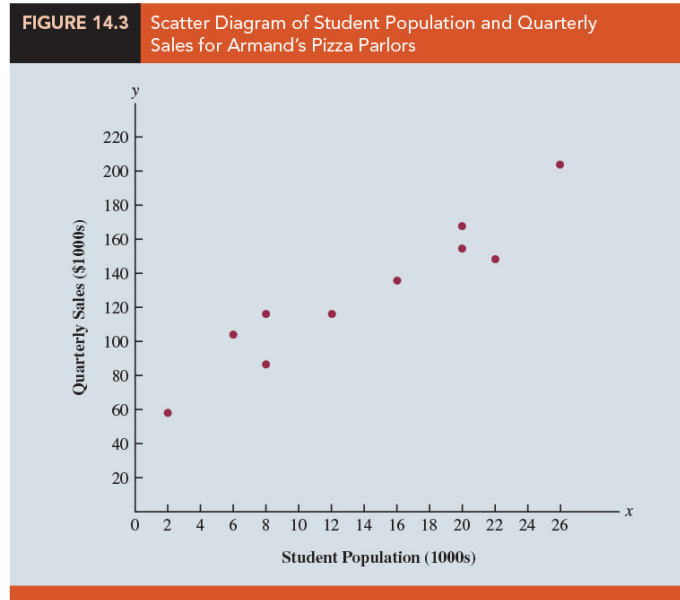
- The least squares method is a procedure for using sample data to find the estimated regression equation.

- (Table 14.1) **Example** Armand's Pizza Parlor

Suppose data were collected from a sample of 10 Armand's Pizza Parlor restaurants located near college campuses. For the i th observation or restaurant in the sample, x_i is the size of the student population (in thousands) and y_i is the quarterly sales (in thousands of dollars).

Restaurant i	Student Population (1000s) x_i	Quarterly Sales (\$1000s) y_i
1	2	58
2	6	105
3	8	88
4	8	118
5	12	117
6	16	137
7	20	157
8	20	169
9	22	149
10	26	202

3. (Figure 14.3) Scatter diagrams for regression analysis are constructed with the independent variable x (student population) on the horizontal axis and the dependent variable y (quarterly sales) on the vertical axis.



- (a) The scatter diagram enables us to observe the data graphically and to draw preliminary conclusions about the possible relationship between the variables.
- (b) Quarterly sales appear to be higher at campuses with larger student populations.
- (c) In addition, for these data the relationship between the size of the student population and quarterly sales appears to be approximated by a straight line.
- (d) A positive linear relationship is indicated between x and y .
- (e) We therefore choose the simple linear regression model to represent the relationship between quarterly sales and student population.
- (f) Next task is to use the sample data in Table 14.1 to determine the values of b_0 and b_1 in the estimated simple linear regression equation.
4. For the i th restaurant, the estimated regression equation provides

$$\hat{y}_i = b_0 + b_1 x_i \quad (14.4)$$

where

- \hat{y}_i : predicted value of quarterly sales (\$1000s) for the i th restaurant
 - b_0 : the y-intercept of the estimated regression line
 - b_1 : the slope of the estimated regression line
 - x_i : size of the student population (1000s) for the i th restaurant
5. In simple linear regression, each observation (x_i, y_i) consists of two values: one for the independent variable and one for the dependent variable.
 6. Every restaurant in the sample will have an observed value of sales y_i and a predicted value of sales \hat{y}_i .
 7. For the estimated regression line to provide a good fit to the data, we want the differences between the observed sales values and the predicted sales values to be small.
 8. **(the least squares method)** The least squares method uses the sample data to provide the values of b_0 and b_1 that minimize the sum of the squares of the deviations between the observed values of the dependent variable y_i and the predicted values of the dependent variable \hat{y}_i .

9. Least Squares Criterion

$$\min \sum_{i=1}^n (y_i - \hat{y}_i)^2 \quad (14.5)$$

where

- y_i : observed value of the dependent variable for the i th observation
- \hat{y}_i : predicted value of the dependent variable for the i th observation

10. **Slope and Y-Intercept for the Estimated Regression Equation** Differential calculus can be used to show (see Appendix 14.1) that the values of b_0 and b_1 that minimize expression (14.5) can be found by:

$$b_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum x_i^2 - n\bar{x}^2} \quad (14.6)$$

$$b_0 = \frac{\bar{y} - b_1 \bar{x}}{\quad} \quad (14.7)$$

where

- x_i : value of the independent variable for the i th observation
- y_i : value of the dependent variable for the i th observation
- \bar{x} : mean value for the independent variable
- \bar{y} : mean value for the dependent variable
- n : total number of observations

補充說明：

 Question (p660)

Using data in Table 14.2 to calculate the slope and intercept of the estimated regression equation for Armand's Pizza Parlors example.

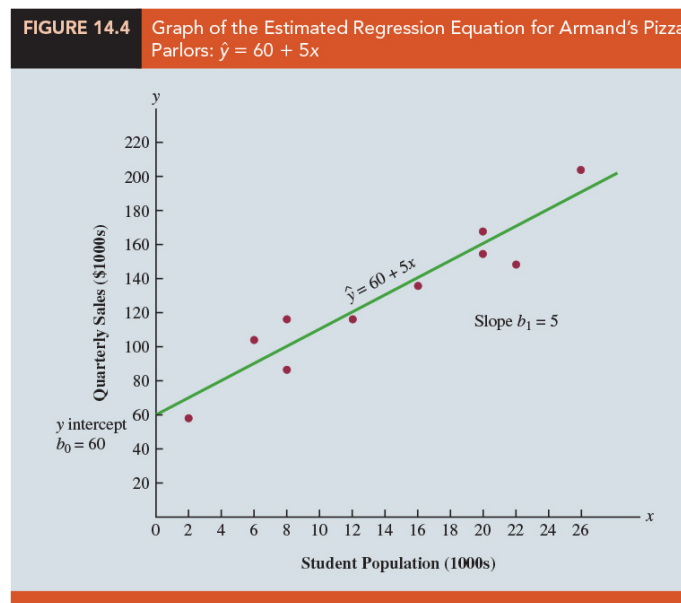
sol:

$$\begin{aligned}\bar{x} &= \frac{\sum x_i}{n} = \frac{140}{10} = 14, & \bar{y} &= \frac{\sum y_i}{n} = \frac{1300}{10} = 130 \\ b_1 &= \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \frac{2840}{568} = 5 \\ b_0 &= \bar{y} - b_1 \bar{x} = 130 - 5(14) = 60\end{aligned}$$

Thus, the estimated regression equation is $\hat{y} = 60 + 5x$.

Restaurant i	x_i	y_i	$x_i - \bar{x}$	$y_i - \bar{y}$	$(x_i - \bar{x})(y_i - \bar{y})$	$(x_i - \bar{x})^2$
1	2	58	-12	-72	864	144
2	6	105	-8	-25	200	64
3	8	88	-6	-42	252	36
4	8	118	-6	-12	72	36
5	12	117	-2	-13	26	4
6	16	137	2	7	14	4
7	20	157	6	27	162	36
8	20	169	6	39	234	36
9	22	149	8	19	152	64
10	26	202	12	72	864	144
Totals	140	1300			2840	568
	Σx_i	Σy_i			$\Sigma(x_i - \bar{x})(y_i - \bar{y})$	$\Sigma(x_i - \bar{x})^2$

11. (Figure 14.4) The graph of this equation on the scatter diagram.



- (a) The slope of the estimated regression equation $(b_1 = 5)$ is positive, implying that as student population increases, sales increase.
- (b) We can conclude (based on sales measured in \$1000s and student population in 1000s) that an increase in the student population of 1000 is associated with an increase of \$5000 in (mean) expected sales; that is, quarterly sales are expected to increase by \$5 per student.

12. If we wanted to predict quarterly sales for a restaurant to be located near a campus with 16,000 students, we would compute

$$\hat{y} = \underline{60 + 5(16) = 140}$$

Hence, we would predict quarterly sales of \$140,000 for this restaurant.

13. This least squares criterion is used to choose the equation that provides the best fit.
14. If some other criterion were used, such as minimizing the sum of the absolute deviations between y_i and \hat{y}_i , a different equation would be obtained. In practice, the least squares method is the most widely used.

14.3 Coefficient of Determination

- How well does the estimated regression equation fit the data? The coefficient of determination provides a measure of the goodness of fit for the estimated regression equation.
- (residual)** For the i th observation, the difference between the observed value of the dependent variable, y_i , and the predicted value of the dependent variable, \hat{y}_i , is called the i th residual. The i th residual represents the error in using \hat{y}_i to estimate y_i . Thus, for the i th observation, the residual is $y_i - \hat{y}_i$.
- (Sum of Squares Due to Error)** The sum of squares of these residuals or errors is the quantity that is minimized by the least squares method. This quantity, also known as the sum of squares due to error, is denoted by SSE :

$$SSE = \underline{\sum (y_i - \hat{y}_i)^2} \quad (14.8)$$

- (Table 14.3) The value of SSE is a measure of the error in using the estimated regression equation to predict the values of the dependent variable in the sample. $SSE = 1530$ measures the error in using the estimated regression equation $\hat{y} = 60 + 5x$ to predict sales.

TABLE 14.3 Calculation of SSE for Armand's Pizza Parlors

Restaurant i	$x_i =$ Student Population (1000s)	$y_i =$ Quarterly Sales (\$1000s)	Predicted Sales $\hat{y}_i = 60 + 5x_i$	Error $y_i - \hat{y}_i$	Squared Error $(y_i - \hat{y}_i)^2$
1	2	58	70	-12	144
2	6	105	90	15	225
3	8	88	100	-12	144
4	8	118	100	18	324
5	12	117	120	-3	9
6	16	137	140	-3	9
7	20	157	160	-3	9
8	20	169	160	9	81
9	22	149	170	-21	441
10	26	202	190	12	144
					SSE = 1530

5. Now suppose we are asked to develop an estimate of quarterly sales without knowledge of the size of the student population. Without knowledge of any related variables, we would use the sample mean as an estimate of quarterly sales at any given restaurant.
6. (Table 14.4) (**Total Sum of Squares**) We show the sum of squared deviations obtained by using the sample mean $\bar{y} = 130$ to predict the value of quarterly sales for each restaurant in the sample. For the i th restaurant in the sample, the difference $y_i - \bar{y}$ provides a measure of the error involved in using \bar{y} to predict sales. The corresponding sum of squares, called the total sum of squares, is denoted SST .

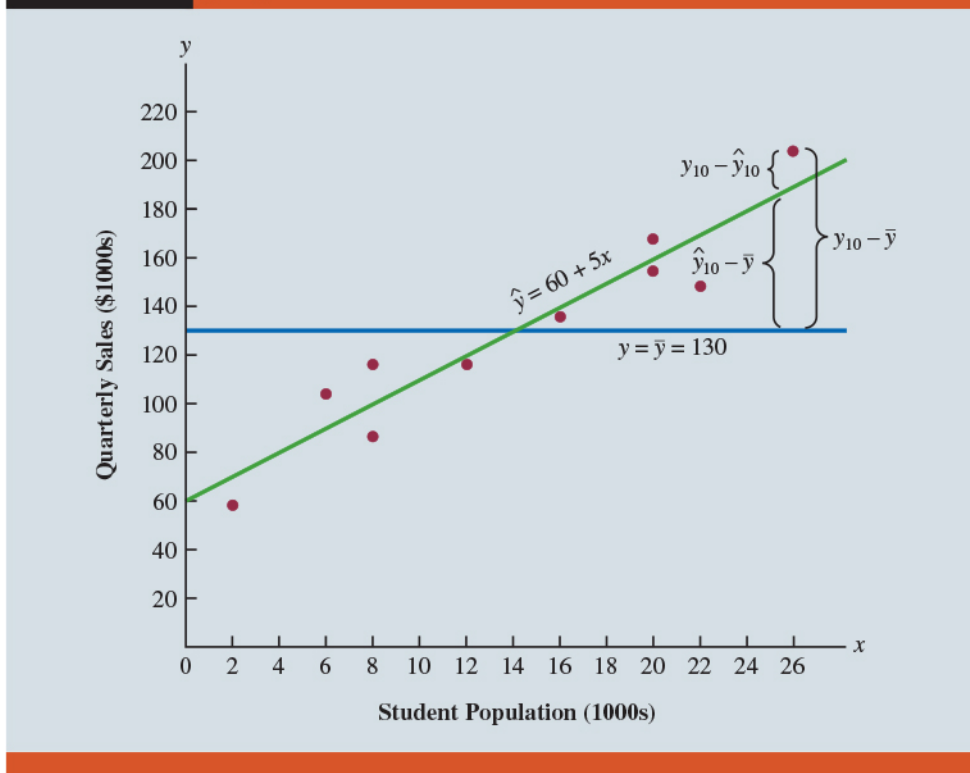
$$SST = \underline{\sum (y_i - \bar{y})^2} \quad (14.9)$$

TABLE 14.4 Computation of the Total Sum of Squares for Armand's Pizza Parlors

Restaurant i	$x_i =$ Student Population (1000s)	$y_i =$ Quarterly Sales (\$1000s)	Deviation $y_i - \bar{y}$	Squared Deviation $(y_i - \bar{y})^2$
1	2	58	-72	5184
2	6	105	-25	625
3	8	88	-42	1764
4	8	118	-12	144
5	12	117	-13	169
6	16	137	7	49
7	20	157	27	729
8	20	169	39	1521
9	22	149	19	361
10	26	202	72	5184
				SST = 15,730

7. (Figure 14.5)

FIGURE 14.5 Deviations About the Estimated Regression Line and the Line $y = \bar{y}$ for Armand's Pizza Parlors



8. We can think of SST as a measure of how well the observations cluster about

the \bar{y} line and SSE as a measure of how well the observations cluster about the \hat{y} line.

9. (**Sum of Squares Due to Regression**) the sum of squares due to regression, is denoted SSR , measures how much the \hat{y} values on the estimated regression line deviate from \bar{y} :

$$SSR = \underline{\sum(\hat{y}_i - \bar{y})^2} \quad (14.10)$$

10. (**Relationship Among SST , SSR , and SSE**) From the preceding discussion, we should expect that SST , SSR , and SSE are related.

$$\underline{SST = SSR + SSE} \quad (14.11)$$

where

- SST : total sum of squares
- SSR : sum of squares due to regression
- SSE : sum of squares due to error

11. SSR can be thought of as the explained portion of SST , and SSE can be thought of as the unexplained portion of SST .

12. Example Armand's Pizza Parlors example
we already know that $SSE = 1530$ and $SST = 15,730$; therefore, solving for SSR in equation (14.11), we find that the sum of squares due to regression is

$$SSR = \underline{SST - SSE} = 15,730 - 1530 = 14,200$$

13. How the three sums of squares, SST , SSR , and SSE , can be used to provide a measure of the goodness of fit for the estimated regression equation?

- (a) The estimated regression equation would provide a perfect fit if every value of the dependent variable y_i happened to lie on the estimated regression line.
- (b) In this case, $y_i - \hat{y}_i$ would be zero for each observation, resulting in $SSE = 0$.
- (c) Because $SST = SSR + SSE$, we see that for a perfect fit SSR must equal SST , and the ratio ($\frac{SSR}{SST}$) must equal one.

- (d) Poorer fits will result in larger values for SSE . Hence the poorest fit occurs when $SSR = 0$ and $SSE = SST$.
14. (**Coefficient of Determination**) The ratio SSR/SST , which will take values between zero and one, is used to evaluate the goodness of fit for the estimated regression equation. This ratio is called the coefficient of determination and is denoted by (r^2) (Other textbook: R^2).

$$r^2 = \frac{SSR}{SST} \quad (14.12)$$

15. When we express the coefficient of determination as a percentage, r^2 can be interpreted as the percentage of the total sum of squares that can be explained by using the estimated regression equation.
16. (**Example**) Armand's Pizza Parlors example

- (a) The value of the coefficient of determination is

$$r^2 = \frac{SSR}{SST} = \frac{14,200}{15,730} = 0.9027$$

- (b) For Armand's Pizza Parlors, we can conclude that 90.27% of the total sum of squares can be explained by using the estimated regression equation $\hat{y} = 60 + 5x$ to predict quantity
- (c) In other words, 90.27% of the variability in sales can be explained by the linear relationship between the size of the student population and sales. We should be pleased to find such a good fit for the estimated regression equation.

Correlation Coefficient

- In Chapter 3 we introduced the correlation coefficient as a descriptive measure of the strength of linear association between two variables, x and y . Values of the correlation coefficient are always between -1 and $+1$.
- A value of $+1$ indicates that the two variables x and y are perfectly related in a positive linear sense. A value of -1 indicates that x and y are perfectly related in a negative linear sense, with all data points on a straight line that has a negative slope. Values of the correlation coefficient close to zero indicate that x and y are not linearly related.

3. (**Sample Correlation Coefficient**) If a regression analysis has already been performed and the coefficient of determination r^2 computed, the sample correlation coefficient can be computed:

$$r_{xy} = \frac{(\text{sign of } b_1)\sqrt{\text{Coefficient of determination}}}{1} = \frac{(\text{sign of } b_1)\sqrt{r^2}}{1} \quad (14.13)$$

where b_1 is the slope of the estimated regression equation $\hat{y} = b_0 + b_1x$

補充說明 : Show that the coefficient of determination of a simple linear regression is the square of the sample correlation coefficient of $(x_1, y_1), \dots, (x_n, y_n)$.

4. **Example** Armand's Pizza Parlor example
the value of the coefficient of determination corresponding to the estimated regression equation $\hat{y} = 60 + 5x$ is 0.9027. Because the slope of the estimated regression equation is positive, equation (14.13) shows that the sample correlation coefficient is $+\sqrt{0.9027} = +0.9501$. (a strong positive linear association exists between x and y .)
5. In the case of a linear relationship between two variables, both the coefficient of determination and the sample correlation coefficient provide measures of the strength of the relationship. The coefficient of determination provides a measure between zero and one, whereas the sample correlation coefficient provides a measure between -1 and $+1$.

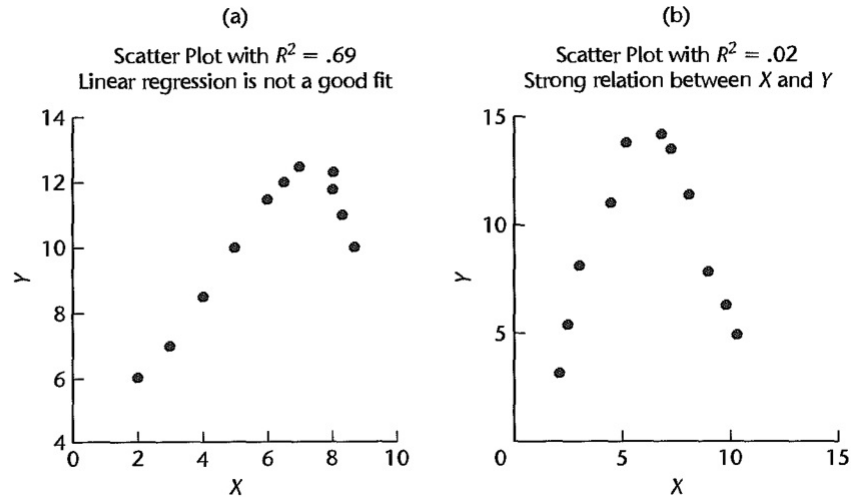
6. Although the sample correlation coefficient is restricted to a linear relationship between two variables, the coefficient of determination can be used for nonlinear relationships and for relationships that have two or more independent variables. Thus, the coefficient of determination provides a wider range of applicability.

(補充) limitations of R^2 : three common misunderstandings

Source : Michael H. Kutner et al. (2019), Applied Linear Statistical Models: Applied Linear Regression Models, Mcgraw-Hill Inc., (5th edition)

1. **Misunderstanding 1:** A high R^2 indicates that useful predictions can be made. (not necessarily correct)
 - (a) (Toluca Company Example) the coefficient of determination was high ($R^2 = 0.82$). Yet the 90 percent prediction interval for the next lot, consisting of 100 units, was wide (332 to 507 hours) and not precise enough to permit management to schedule workers effectively.
 - (b) Misunderstanding 1 arises because R^2 measures only a relative reduction from SST and provides no information about absolute precision for estimating a mean response or predicting a new observation.
2. **Misunderstanding 2:** A high R^2 indicates that the estimated regression line is a good fit. (not necessarily correct)
 - (a) (Figure 2.9a) a scatter plot where R^2 is high ($R^2 = 0.69$). Yet a linear regression function would not be a good fit since the regression relation is curvilinear.
3. **Misunderstanding 3:** A R^2 near zero indicates that X and Y are not related. (not necessarily correct).
 - (a) (Figure 2.9b) a scatter plot where R^2 between X and Y is $R^2 = 0.02$. Yet X and Y are strongly related; however, the relationship between the two variables is curvilinear.
 - (b) Misunderstandings 2 and 3 arise because R^2 measures the degree of linear association between X and Y , whereas the actual regression relation may be curvilinear.

FIGURE 2.9
Illustrations
of Two Misun-
derstandings
about
Coefficient of
Determination.



14.4 Model Assumptions

1. In conducting a regression analysis, we begin by making an assumption about the appropriate model for the relationship between the dependent and independent variable(s).
2. For the case of simple linear regression, the assumed regression model is

$$\underline{y = \beta_0 + \beta_1 x + \epsilon}$$

3. Then the least squares method is used to develop values for b_0 and b_1 , the estimates of the model parameters β_0 and β_1 , respectively. The resulting estimated regression equation is

$$\underline{\hat{y} = b_0 + b_1 x}$$

Even with a large value of r^2 , the estimated regression equation should not be used until further analysis of the appropriateness of the assumed model has been conducted.

4. An important step in determining whether the assumed model is appropriate involves testing for the significance of the relationship. The tests of significance in regression analysis are based on the following assumptions about the error term ϵ .

5. **Assumptions About The Error Term ϵ in the Regression Model**

$$y = \beta_0 + \beta_1 x + \epsilon$$

- (a) The error term ϵ is a random variable with a mean or expected value of zero; that is, $E(\epsilon) = 0$.

Implication: β_0 and β_1 are constants, thus, for a given value of x , the expected value of y is

$$\underline{E(y) = \beta_0 + \beta_1 x} \quad (14.14)$$

As we indicated previously, equation (14.14) is referred to as the regression equation.

- (b) The variance of ϵ , denoted by σ^2 , is the same for all values of x .

Implication: The variance of y about the regression line equals σ^2 and is the same for all values of x .

- (c) The values of ϵ are independent.

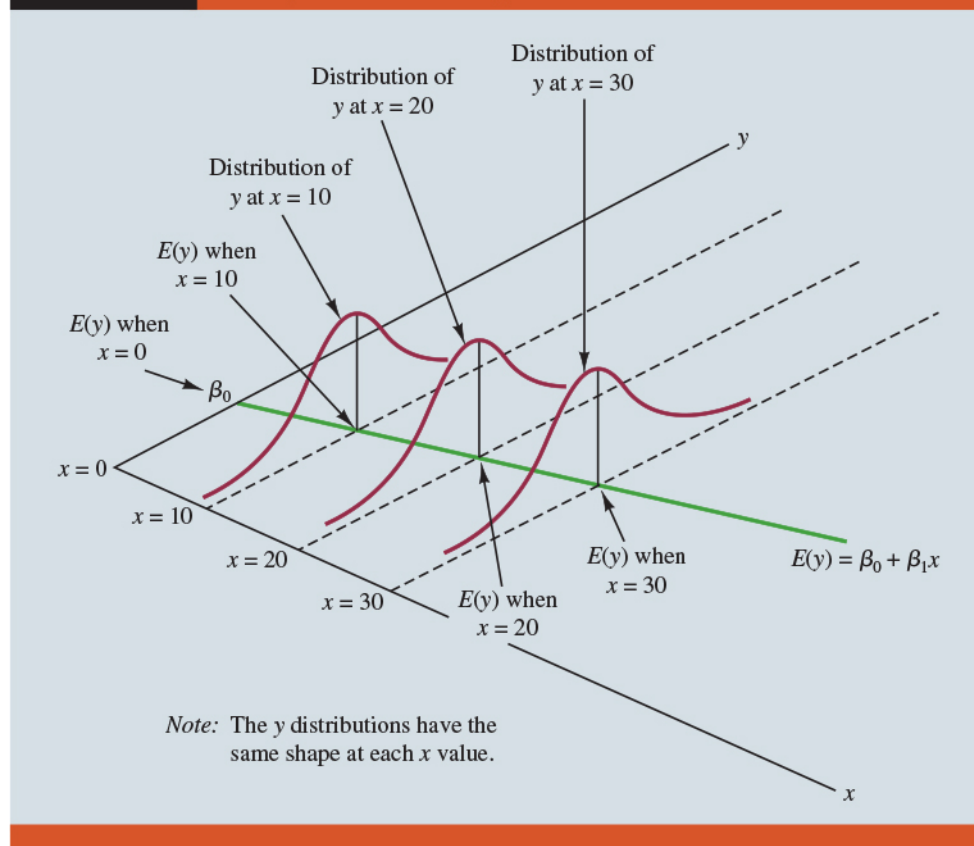
Implication: The value of ϵ for a particular value of x is not related to the value of ϵ for any other value of x ; thus, the value of y for a particular value of x is not related to the value of y for any other value of x .

- (d) The error term ϵ is a normally distributed r.v. for all values of x .

Implication: Because y is a linear function of ϵ , y is also a normally distributed random variable for all values of x .

6. Figure 14.6 illustrates the model assumptions and their implications; note that in this graphical interpretation, the value of $E(y)$ changes according to the specific value of x considered. However, regardless of the x value, the probability distribution of ϵ and hence the probability distributions of y are normally distributed, each with the same variance.

FIGURE 14.6 Assumptions for the Regression Model



7. The specific value of the error ϵ at any particular point depends on whether the actual value of y is greater than or less than $E(y)$.
8. We assume that a straight line represented by $\beta_0 + \beta_1 x$ is the basis for the relationship between the variables.

14.5 Testing for Significance

1. In a simple linear regression equation, the mean or expected value of y is a linear function of x : $E(y) = \beta_0 + \beta_1 x$. If the value of β_1 is zero, the mean value of y does not depend on the value of x and hence we would conclude that x and y are not linearly related.

- To test for a significant regression relationship, we must conduct a hypothesis test to determine whether the value of β_1 is zero.
- Two tests are commonly used. Both require an estimate of σ^2 , the variance of ϵ in the regression model.

Estimate of σ^2

- From the regression model and its assumptions we can conclude that σ^2 , the variance of ϵ , also represents the variance of the y values about the regression line.
- Thus, SSE , the sum of squared residuals, is a measure of the variability of the actual observations about the estimated regression line.

$$SSE = \sum (y_i - \hat{y}_i)^2 = \sum (y_i - b_0 - b_1 x_i)^2$$

- Statisticians have shown that SSE has $n-2$ degrees of freedom because two parameters (β_0 and β_1) must be estimated to compute SSE .
- The **mean square error (MSE)** provides the estimate of σ^2 ; it is SSE divided by its degrees of freedom.

5. Mean Square Error (Estimate of σ^2)

$$s^2 = MSE = \frac{SSE}{n-2} \quad (14.15)$$

6. Standard Error of the Estimate

$$s = \sqrt{MSE} = \sqrt{\frac{SSE}{n-2}} \quad (14.16)$$

- Example** Armand's Pizza Parlors example

$$s^2 = MSE = \frac{1530}{8} = 191.25$$

provides an unbiased estimate of σ^2 .

$$s = \sqrt{MSE} = \sqrt{191.25} = 13.829.$$

***t* Test**

1. The purpose of the *t* test is to see whether we can conclude that $\beta_1 \neq 0$. We will use the sample data to test the following hypotheses about the parameter β_1 .

$$\underline{H_0 : \beta_1 = 0, \quad H_a : \beta_1 \neq 0}$$

2. If H_0 is rejected, we will conclude that $\beta_1 \neq 0$ and that a statistically significant relationship exists between the two variables.
3. If H_0 cannot be rejected, we will have insufficient evidence to conclude that a significant relationship exists.
4. The properties of the sampling distribution of β_1 , the least squares estimator of b_1 , provide the basis for the hypothesis test.

5. Sampling Distribution of b_1

- Expected Value: $E(b_1) = \beta_1$
- Standard Deviation: $\sigma_{b_1} = \frac{\sigma}{\sqrt{\sum(x_i - \bar{x})^2}}$

(証明):

- Distribution Form: Normal (14.17)

6. Because we do not know the value of σ , we develop an estimate of σ_{b_1} , denoted s_{b_1} , by estimating σ with s in equation (14.17). Thus, we obtain the following estimate of σ_{b_1} .

7. Estimated Standard Deviation of b_1

$$s_{b_1} = \frac{s}{\sqrt{\sum(x_i - \bar{x})^2}} \quad (14.18)$$

8. The standard deviation of b_1 is also referred to as the standard error of b_1 . Thus, s_{b_1} provides an estimate of the standard error of b_1 .
9. The t test for a significant relationship is based on the fact that the test statistic

$$\frac{b_1 - \beta_1}{s_{b_1}}$$

follows a t distribution with $n-2$ degrees of freedom. If the null hypothesis is true, then $\beta_1 = 0$ and $t = b_1/s_{b_1}$.

10. t Test for Significance in Simple Linear Regression

(a) Hypothesis:

$$H_0 : \beta_1 = 0, \quad H_a : \beta_1 \neq 0$$


(b) Test Statistic: $t = \frac{b_1}{s_{b_1}}$ (14.19)

(c) Rejection Rule:

i. p -value approach: Reject H_0 if $p\text{-value} \leq \alpha$

ii. Critical value approach: Reject H_0 if $t \leq -t_{\alpha/2}$ or if $t \geq t_{\alpha/2}$.

where $t_{\alpha/2}$ is based on a t distribution with $n-2$ degrees of freedom.

 **Question** (p678)

Conduct t test of significance for Armand's Pizza Parlors at the $\alpha = 0.01$ level of significance.

sol:

- Hypothesis: $H_0 : \beta_1 = 0, \quad H_a : \beta_1 \neq 0$.
- Level of significance: $\alpha = 0.01$.
- Test statistic (under H_0): $t = \frac{b_1}{s_{b_1}} = \frac{5}{0.5803} = 8.62$.
- Decision rule
 - Reject H_0 if $p\text{-value} < \alpha$, or

- (b) Reject H_0 if $|t| \geq t_{\alpha/2, n-2} = t_{0.005, 8} = 3.355$. (Table 2 of Appendix D, upper tail of the t distribution)
5. Decision:
- (a) $p\text{-value (0.000) less than } 2(0.005) = 0.01$. (Software), we reject H_0 .
- (b) $t = 8.62 > t_{\alpha/2, n-2} = 3.355$, we reject H_0 .
6. Conclusion: We reject H_0 and conclude that β_1 is not equal to zero .
This evidence is sufficient to conclude that a significant relationship exists between student population and quarterly sales .

Confidence Interval for β_1

1. The form of a confidence interval for β_1 is as follows:

$$\underline{b_1 \pm t_{\alpha/2, n-2} s_{b_1}}$$

(証明:)

2. The point estimator is b_1 and the margin of error is $t_{\alpha/2} s_{b_1}$.
3. Develop a 99% confidence interval estimate of b_1 for Armand's Pizza Parlors. From Table 2 of Appendix B we find $t_{0.005, 8} = 3.355$. Thus, the 99% confidence interval estimate of b_1 is

$$b_1 \pm t_{\alpha/2, n-2} s_{b_1} = \underline{5 \pm 3.355(0.5803)} = 5 \pm 1.95$$

or 3.05 to 6.95.

4. At the $\alpha = 0.01$ level of significance, we can use the 99% confidence interval as an alternative for drawing the hypothesis testing conclusion for the Armand's data.

- Because 0, the hypothesized value of b_1 , is not included in the confidence interval (3.05 to 6.95), we can reject H_0 and conclude that a significant statistical relationship exists between the size of the student population and quarterly sales.
- In general, a confidence interval can be used to test any two-sided hypothesis about β_1 . If the hypothesized value of β_1 is contained in the confidence interval, do not reject H_0 . Otherwise, reject H_0 .

F Test

- Recall: $SST = \sum(y_i - \bar{y})^2$, $SSE = \sum(y_i - \hat{y}_i)^2$, $SSR = \sum(\hat{y}_i - \bar{y})^2$.
- An F test, based on the F probability distribution, can also be used to test for significance in regression. With only one independent variable, the F test will provide the same conclusion as the t test.
- But with more than one independent variable, only the F test can be used to test for an overall significant relationship.
- If the null hypothesis $H_0 : \beta_1 = 0$ is true, the mean square due to regression (mean square regression), and is denoted MSR . In general,

$$MSR = \frac{SSR}{\text{Regression degrees of freedom}}$$

- The regression degrees of freedom is always equal to the number of independent variables in the model. Because we consider only regression models with one independent variable in this chapter, we have $MSR = SSR/1 = SSR$.

(証明:)

6. If the null hypothesis ($H_0 : \beta_1 = 0$) is true, MSR and MSE are two independent estimates of σ^2 and the sampling distribution of MSR/MSE follows an F distribution with numerator degrees of freedom equal to one and denominator degrees of freedom equal to $n-2$. Therefore, when $\beta_1 = 0$, the value of MSR/MSE should be close to one.
7. If the null hypothesis is false ($\beta_1 \neq 0$), MSR will overestimate σ^2 and the value of MSR/MSE will be inflated; thus, large values of MSR/MSE lead to the rejection of H_0 and the conclusion that the relationship between x and y is statistically significant.

$$MSE = \frac{\sum (y_i - \hat{y}_i)^2}{n-2} = \frac{SSE}{n-2}, \quad MSR = \frac{\sum (\hat{y}_i - \bar{y})^2}{1} = \frac{SSR}{1}.$$

$$E(MSE) = \underline{\sigma^2}, \quad E(MSR) = \underline{\sigma^2 + \beta_1^2 \sum_{i=1}^n (X_i - \bar{X})^2}.$$

(証明:)

8. If H_0 is false, MSE still provides an unbiased estimate of σ^2 and MSR overestimates σ^2 . If H_0 is true, both MSE and MSR provide unbiased estimates of σ^2 ; in this case the value of MSR/MSE should be close to 1.

9. F Test for Significance in Simple Linear Regression

(a) Hypothesis: $H_0 : \beta_1 = 0$, $H_a : \beta_1 \neq 0$

(b) Test Statistic: $F = \frac{MSR}{MSE}$ (14.21)


(c) Rejection Rule:

i. p-value approach: Reject H_0 if $p\text{-value} \leq \alpha$

ii. Critical value approach: Reject H_0 if $F \geq F_{\alpha, 1, n-2}$

where F_α is based on an F distribution with 1 degree of freedom in the numerator and $n-2$ degrees of freedom in the denominator.

10. Decision and Conclusion.

 Question (p680)

Conduct the F test for the Armand's Pizza Parlors example. ($\alpha = 0.01$)

sol:

10. A similar ANOVA table can be used to summarize the results of the F test for significance in regression.

11. (Table 14.5)

TABLE 14.5 General Form of the Anova Table for Simple Linear Regression					
Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F	p -value
Regression	SSR	1	$MSR = \frac{SSR}{1}$	$F = \frac{MSR}{MSE}$	
Error	SSE	$n - 2$	$MSE = \frac{SSE}{n - 2}$		
Total	SST	$n - 1$			

12. (Table 14.6) ANOVA table with the F test computations performed for Armand's Pizza Parlors.

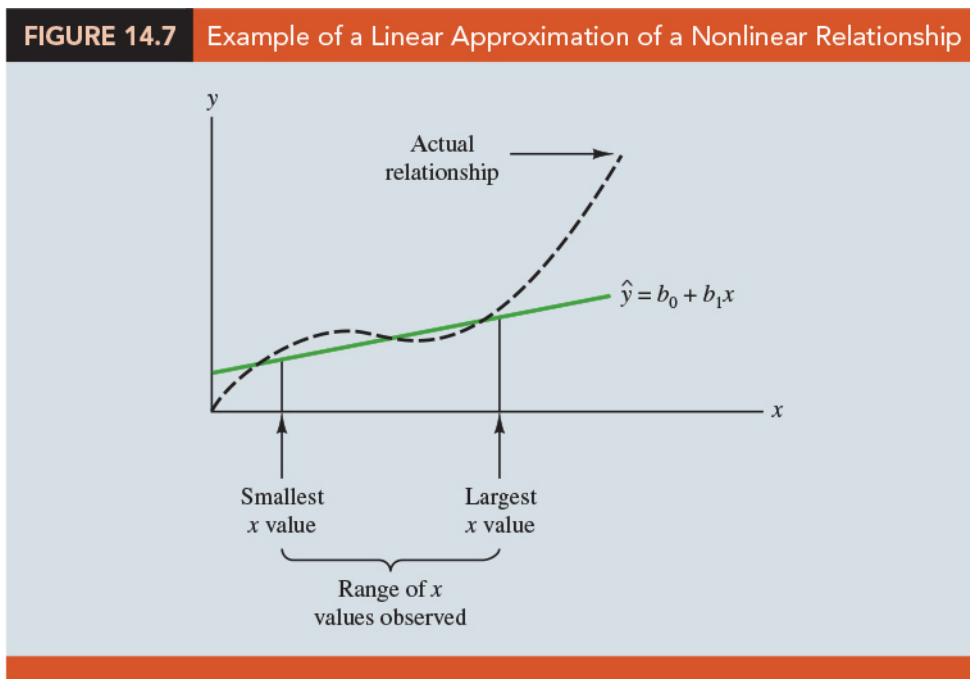
TABLE 14.6 Anova Table for the Armand's Pizza Parlors Problem

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F	p-value
Regression	14,200	1	$\frac{14,200}{1} = 14,200$	$\frac{14,200}{191.25} = 74.25$.000
Error	1530	8	$\frac{1530}{8} = 191.25$		
Total	15,730	9			

Some Cautions About the Interpretation of Significance Tests

1. Rejecting the null hypothesis $H_0 : \beta_1 = 0$ and concluding that the relationship between x and y is significant does not enable us to conclude that a cause-and-effect relationship is present between x and y .
2. Concluding a cause-and-effect relationship is warranted only if the analyst can provide some type of theoretical justification that the relationship is in fact causal.
3. In the Armand's Pizza Parlors example, we can conclude that there is a significant relationship between the size of the student population x and quarterly sales y ; moreover, the estimated regression equation $\hat{y} = 60 + 5x$ provides the least squares estimate of the relationship. We cannot, however, conclude that changes in student population x cause changes in quarterly sales y just because we identified a statistically significant relationship.
4. Armand's managers felt that increases in the student population were a likely cause of increased quarterly sales. Thus, the result of the significance test enabled them to conclude that a cause-and-effect relationship was present.
5. We can state only that x and y are related and that a linear relationship explains a significant portion of the variability in y over the range of values for x observed in the sample.
6. (Figure 14.7) illustrates this situation. The test for significance calls for the rejection of the null hypothesis $H_0 : \beta_1 = 0$ and leads to the conclusion that x and y are

significantly related, but the figure shows that the actual relationship between x and y is not linear.



7. Although the linear approximation provided by $\hat{y} = b_0 + b_1x$ is good over the range of x values observed in the sample, it becomes poor for x values outside that range.
8. Given a significant relationship, we should feel confident in using the estimated regression equation for predictions corresponding to x values within the range of the x values observed in the sample.
9. For Armand's Pizza Parlors, this range corresponds to values of x between 2 and 26. Unless other reasons indicate that the model is valid beyond this range, predictions outside the range of the independent variable should be made with caution.

14.6 Using the Estimated Regression Equation for Estimation and Prediction

1. When using the simple linear regression model, we are making an assumption about the relationship between x and y . We then use the least squares method to obtain the estimated simple linear regression equation.
2. If a significant relationship exists between x and y and the coefficient of determination shows that the fit is good, the estimated regression equation should be useful for estimation and prediction.
3. Example Armand's Pizza Parlors example

- (a) The estimated regression equation is $\hat{y} = 60 + 5x$. \hat{y} can be used as a point estimator of $E(y)$, the mean or expected value of y for a given value of x , and as a predictor of an individual value of y .
- (b) For example, a point estimate of the mean quarterly sales for all restaurant locations near campuses with $x = 10$ (10,000 students) students is

$$\hat{y} = \underline{60 + 5(10) = 110} \quad (\$110,000).$$

In this case we are using \hat{y} as the point estimator of the mean value of y when $x = 10$.

- (c) For example, to predict quarterly sales for a new restaurant Armand's is considering building near Talbot College, a campus with 10,000 students, we would compute

$$\hat{y} = \underline{60 + 5(10) = 110}.$$

Hence, we would predict quarterly sales of \$110,000 for such a new restaurant. In this case, we are using \hat{y} as the predictor of y for a new observation when $x = 10$.

4. Notations:

- (a) x^* = the given value of the independent variable x
- (b) y^* = the random variable denoting the possible values of the dependent variable y when $x = x^*$

(c) $\underline{E(y^*)}$ = the mean or expected value of the dependent variable y when $x = x^*$

(d) $\underline{\hat{y}^* = \beta_0 + \beta_1 x^*}$ = the point estimator of $E(y^*)$ and the predictor of an individual value of y^* when $x = x^*$

5. **Example** Armand's Pizza Parlors example

(a) To illustrate the use of this notation, suppose we want to estimate the mean value of quarterly sales for all Armand's restaurants located near a campus with 10,000 students.

(b) For this case, $\underline{x^* = 10}$ and $\underline{E(y^*)}$ denotes the unknown mean value of quarterly sales for all restaurants where $x^* = 10$.

(c) Thus, the point estimate of $E(y^*)$ is provided by $\underline{\hat{y}^* = 60 + 5(10) = 110}$, or \$110,000.

(d) But, using this notation, $\hat{y}^* = 110$ is also the predictor of quarterly sales for the new restaurant located near Talbot College, a school with 10,000 students.

Interval Estimation

- Point estimators and predictors do not provide any information about the precision associated with the estimate and/or prediction. For that we must develop confidence intervals and prediction intervals.
 - A confidence interval is an interval estimate of the mean value of y for a given value of x .
 - A prediction interval is used whenever we want to predict an individual value of y for a new observation corresponding to a given value of x .
- Although the predictor of y for a given value of x is the same as the point estimator of the mean value of y for a given value of x , the interval estimates we obtain for the two cases are different.
- The margin of error is larger for a prediction interval.

4. Prediction intervals resemble confidence intervals. However, they differ conceptually. A confidence interval represents an inference on a parameter and is an interval that is intended to cover the value of the parameter. A prediction interval is a statement about the value to be taken by a random variable, the new observation y_{new}^* .

Confidence Interval for the Mean Value of y

1. In general, we cannot expect \hat{y}^* to equal $E(y^*)$ exactly. If we want to make an inference about how close \hat{y}^* is to the true mean value $E(y^*)$, we will have to estimate the variance of \hat{y}^* .
2. The formula for estimating the variance of \hat{y}^* , denoted by $s_{\hat{y}^*}^2$, is

$$s_{\hat{y}^*}^2 = \frac{s^2 \left[\frac{1}{n} + \frac{(x^* - \bar{x})^2}{\sum (x_i - \bar{x})^2} \right]}{\quad} \quad (14.22)$$

where $s^2 = \frac{MSE = \sum (y_i - \hat{y}_i)^2}{(n-2)}$.

3. The estimate of the standard deviation of \hat{y}^* is given by the square root of equation (14.22).

$$s_{\hat{y}^*} = s \sqrt{\left[\frac{1}{n} + \frac{(x^* - \bar{x})^2}{\sum (x_i - \bar{x})^2} \right]} \quad (14.23)$$

4. NOTE:

$$y_i = \beta_0^* + \beta_1(x_i - \bar{x}) + \epsilon_i, \beta_0^* = \beta_0 + \beta_1\bar{x} \quad (\text{alternative model})$$

$$\hat{y}^* = b_0^* + b_1(x^* - \bar{x}), b_0^* = b_0 + b_1\bar{x} = \bar{y}$$

$$\hat{y}^* = \bar{y} + b_1(x^* - \bar{x})$$

$$E(\hat{y}^*) = E(y^*)$$

$$\begin{aligned} \sigma_{\hat{y}^*}^2 = Var(\hat{y}^*) &= Var(\bar{y} + b_1(x^* - \bar{x})) \\ &= Var(\bar{y}) + Var(b_1(x^* - \bar{x})) \\ &= \frac{\sigma^2}{n} + (x^* - \bar{x})^2 \frac{\sigma^2}{\sum (x_i - \bar{x})^2} \\ &= \sigma^2 \left[\frac{1}{n} + \frac{(x^* - \bar{x})^2}{\sum (x_i - \bar{x})^2} \right]. \end{aligned}$$

5. **Example** Armand's Pizza Parlors

$s = 13.829$. With $x^* = 10$, $\bar{x} = 14$, and $\sum(x_i - \bar{x})^2 = 568$, we can use equation (14.23) to obtain

$$s_{\hat{y}^*} = \frac{13.829 \sqrt{\frac{1}{10} + \frac{(10 - 14)^2}{568}}}{1} = 13.829 \sqrt{0.1282} = 4.95$$

6. **Theorem:**

$$\frac{\hat{y}^* - E(y^*)}{s_{\hat{y}^*}} \sim t_{(n-2)}$$

7. **Confidence Interval for $E(y^*)$**

$$\hat{y}^* \pm t_{\alpha/2} s_{\hat{y}^*} \quad (14.24)$$

where the confidence coefficient is $1 - \alpha$ and $t_{\alpha/2}$ is based on the t distribution with $(n-2)$ degrees of freedom.

8. **Example** Armand's Pizza Parlors

(a) Develop a 95% confidence interval of the mean quarterly sales for all Armand's restaurants located near campuses with 10,000 students.

(b) We have $t_{0.025,8} = 2.306$. Thus, with $\hat{y}^* = 110$ and a margin of error of $t_{\alpha/2} s_{\hat{y}^*} = 2.306(4.95) = 11.415$, the 95% confidence interval estimate is 110 ± 11.415 .

(c) In dollars, the 95% confidence interval for the mean quarterly sales of all restaurants near campuses with 10,000 students is $\$110,000 \pm \$11,415$. Therefore, the 95% confidence interval for the mean quarterly sales when the student population is 10,000 is $\$98,585$ to $\$121,415$.

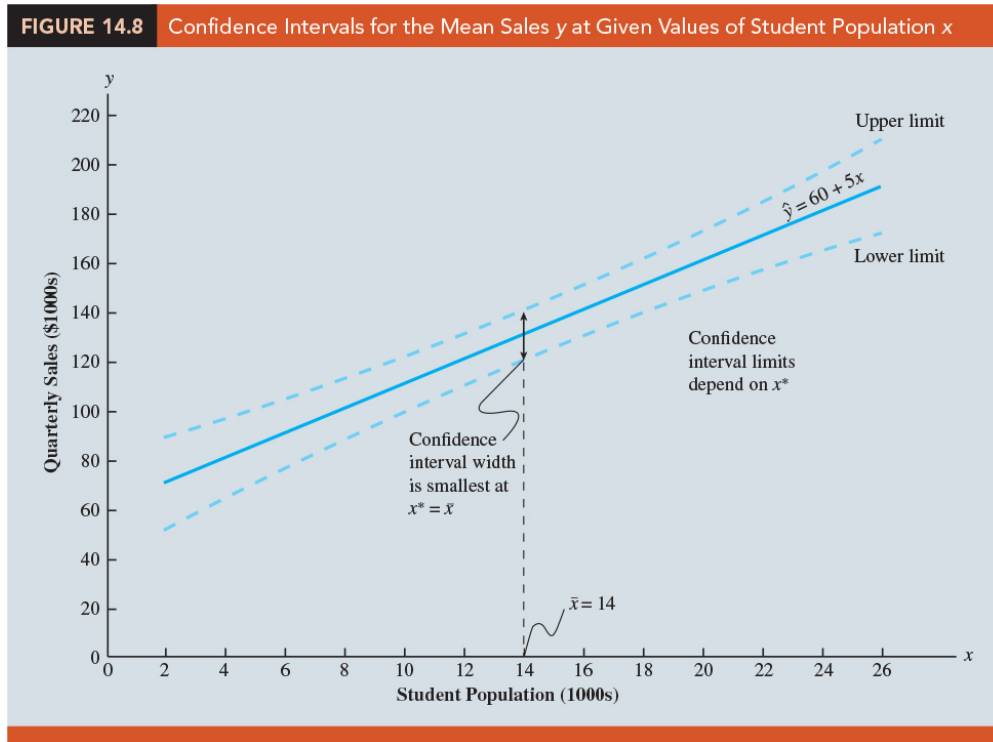
9. Note that the estimated standard deviation of \hat{y}^* given by equation (14.23) is smallest when $x^* - \bar{x} = 0$.

10. In this case the estimated standard deviation of \hat{y}^* becomes

$$s_{\hat{y}^*} = s \sqrt{\left[\frac{1}{n} + \frac{(x^* - \bar{x})^2}{\sum(x_i - \bar{x})^2} \right]} = s \sqrt{\frac{1}{n}}$$

This result implies that we can make the best or most precise estimate of the mean value of y whenever $x^* = \bar{x}$.

11. (Figure 14.8) the further x^* is from \bar{x} , the larger $x^* - \bar{x}$ becomes. As a result, the confidence interval for the mean value of y will become wider as x^* deviates more from \bar{x} .



Prediction Interval for an Individual Value of y

1. The predictor of y^* , the value of y corresponding to the given x^* , is $\hat{y}^* = \beta_0 + \beta_1 x^*$.
2. For the new restaurant located near Talbot College, $x^* = 10$ and the prediction of quarterly sales is $\hat{y}^* = 60 + 5(10) = 110$, or \$110,000. Note that the prediction of quarterly sales for the new Armand's restaurant near Talbot College is the same as the point estimate of the mean sales for all Armand's restaurants located near campuses with 10,000 students.
3. Determine the variance associated with using \hat{y}^* as a predictor of y when $x = x^*$. This variance is made up of the sum of the following two components.
 - (a) The (estimated) variance of the y^* values about the mean $E(y^*)$: s^2 .
 - (b) The (estimated) variance associated with using \hat{y}^* to estimate $E(y^*)$: $s_{\hat{y}^*}^2$.

4. The formula for estimating the variance corresponding to the prediction of the value of y when $x = x^*$, denoted s_{pred}^2 , is

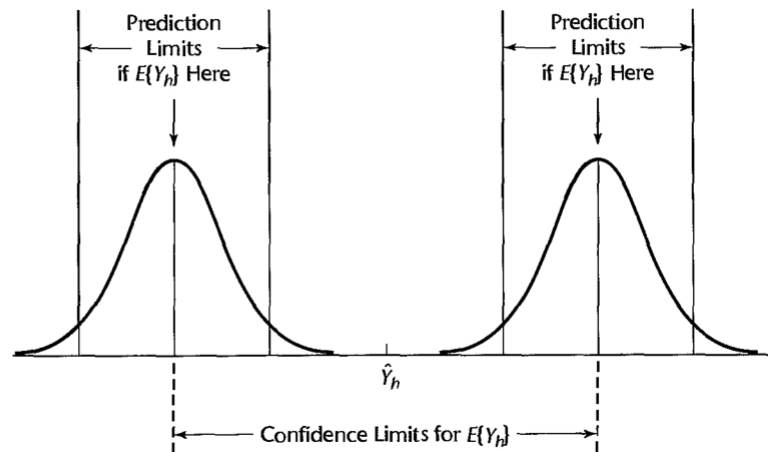
$$\begin{aligned}
 s_{pred}^2 &= \frac{s^2 + s_{\hat{y}^*}^2}{1} \\
 &= \frac{s^2 + s^2 \left[\frac{1}{n} + \frac{(x^* - \bar{x})^2}{\sum (x_i - \bar{x})^2} \right]}{1} \\
 &= \frac{s^2 \left[1 + \frac{1}{n} + \frac{(x^* - \bar{x})^2}{\sum (x_i - \bar{x})^2} \right]}{1} \quad (14.25)
 \end{aligned}$$

5. Theorem:

$$\frac{\hat{y}^* - y_{new}^*}{s_{pred}} \sim t_{(n-2)}$$

$$\begin{aligned}
 \sigma_{pred}^2 &= Var(\hat{y}^* - y_{new}^*) \\
 &= Var(\hat{y}^*) + Var(y_{new}^*) \\
 &= Var(\hat{y}^*) + \sigma^2
 \end{aligned}$$

FIGURE 2.5
Prediction of $Y_{h(new)}$ when Parameters Unknown.



6. (Armand's Pizza Parlors) the estimated standard deviation corresponding to the prediction of quarterly sales for a new restaurant located near Talbot College, a campus with 10,000 students, is computed as follows.

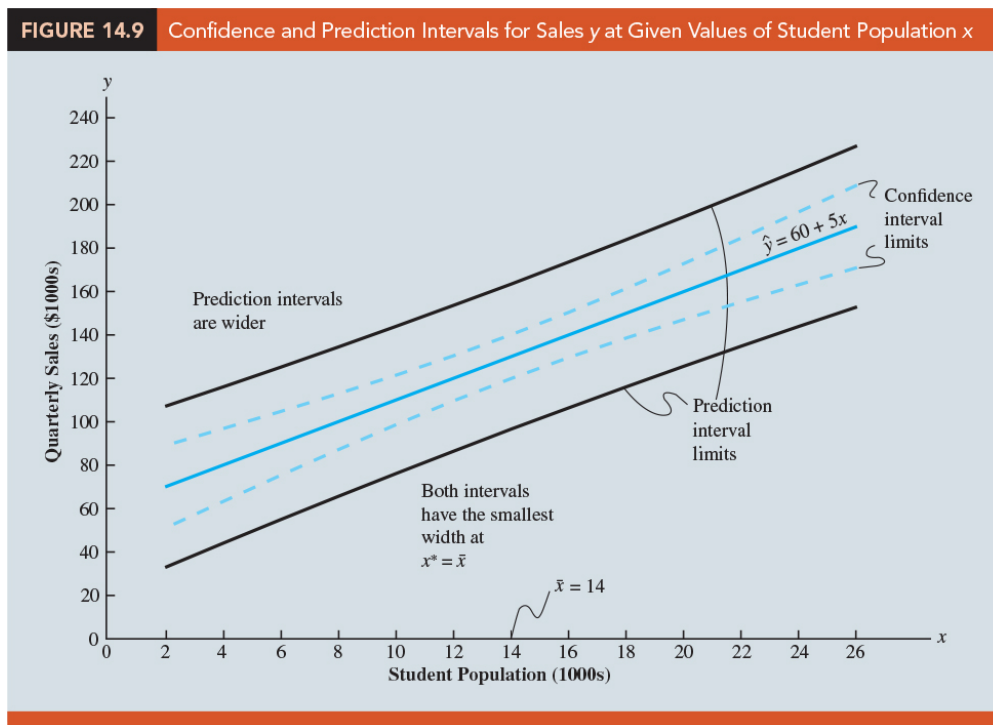
$$s_{pred} = \frac{13.829 \sqrt{1 + \frac{1}{10} + \frac{(10 - 14)^2}{568}}}{1} = 13.829 \sqrt{1.282} = 14.69$$

7. Prediction Interval For y^*

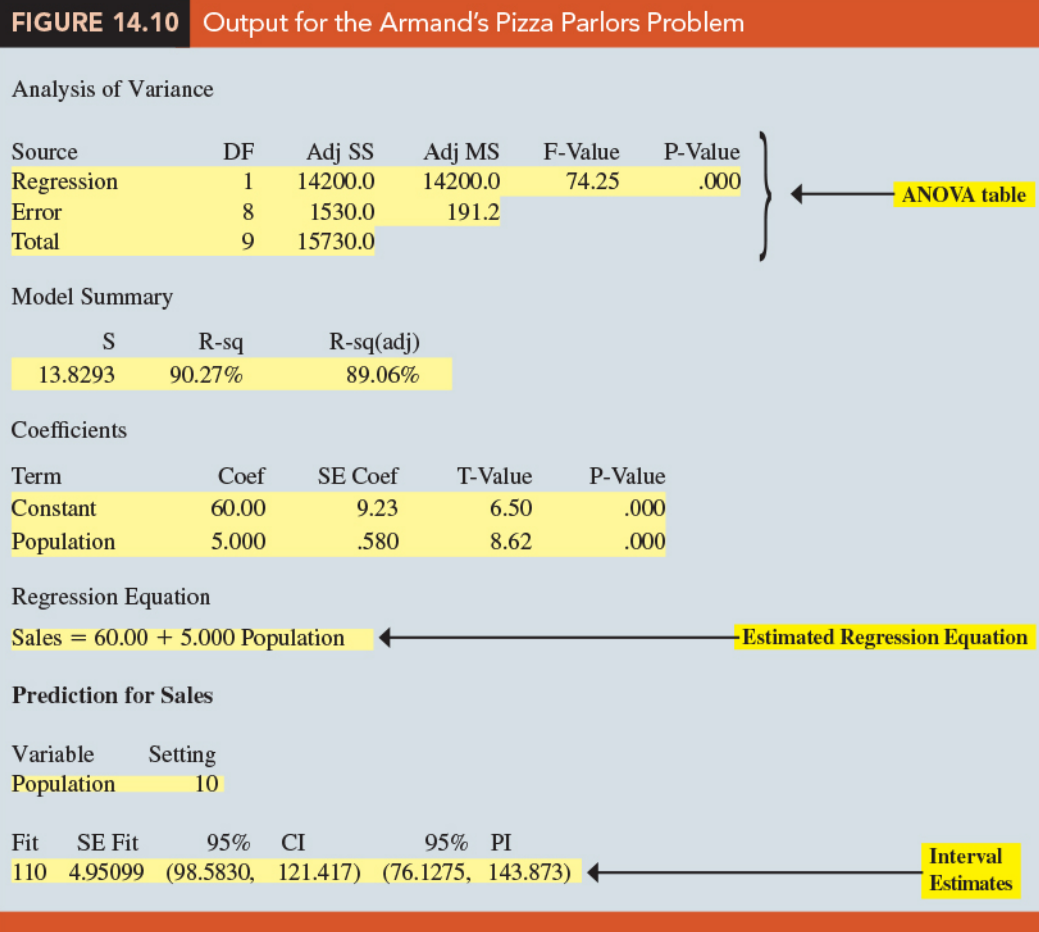
$$\hat{y}^* \pm t_{\alpha/2} s_{pred} \quad (14.27)$$

where the confidence coefficient is $1-\alpha$ and $t_{\alpha/2}$ is based on the t distribution with $n-2$ degrees of freedom.

8. (Armand's Pizza Parlors) The 95% prediction interval for quarterly sales for the new Armand's restaurant located near Talbot College, with $\hat{y}^* = 110$ and a margin of error of $t_{0.025} s_{pred} = 2.306(14.69) = 33.875$, the 95% prediction interval is 110 ± 33.875 (\$76,125 to \$143,875).
9. Note that the prediction interval for the new restaurant located near Talbot College, a campus with 10,000 students, is wider than the confidence interval for the mean quarterly sales of all restaurants located near campuses with 10,000 students. The difference reflects the fact that we are able to estimate the mean value of y more precisely than we can predict an individual value of y .
10. (Figure 14.9) Confidence intervals and prediction intervals are both more precise when the value of the independent variable x^* is closer to \bar{x} .



14.7 Computer Solution



14.8 Residual Analysis: Validating Model Assumptions

1. Residual for observation i : the difference between the observed value of the dependent variable (y_i) and the predicted value of the dependent variable (\hat{y}_i), $y_i - \hat{y}_i$.
2. An analysis of the corresponding residuals will help determine whether the assumptions made about the regression model are appropriate.

3. (Table 14.7)

Student Population x_i	Sales y_i	Predicted Sales $\hat{y}_i = 60 + 5x_i$	Residuals $y_i - \hat{y}_i$
2	58	70	-12
6	105	90	15
8	88	100	-12
8	118	100	18
12	117	120	-3
16	137	140	-3
20	157	160	-3
20	169	160	9
22	149	170	-21
26	202	190	12

4. **Example** Armand's Pizza Parlors

(a) A simple linear regression model was assumed.

$$y = \beta_0 + \beta_1 x + \epsilon \quad (14.29)$$

This model indicates that we assumed quarterly sales (y) to be a linear function of the size of the student population (x) plus an error term ϵ . In Section 14.4 we made the following assumptions about the error term ϵ .

1. $E(\epsilon) = 0$.
2. The variance of ϵ is the same for all values of x . $Var(\epsilon) = \sigma^2$.
3. The values of ϵ are independent.
4. The error term ϵ has a normal distribution.

- (b) These assumptions provide the theoretical basis for the t test and the F test used to determine whether the relationship between x and y is significant, and for the confidence and prediction interval estimates presented in Section 14.6.
5. If the assumptions about the error term ϵ appear questionable, the hypothesis tests about the significance of the regression relationship and the interval estimation results may not be valid.

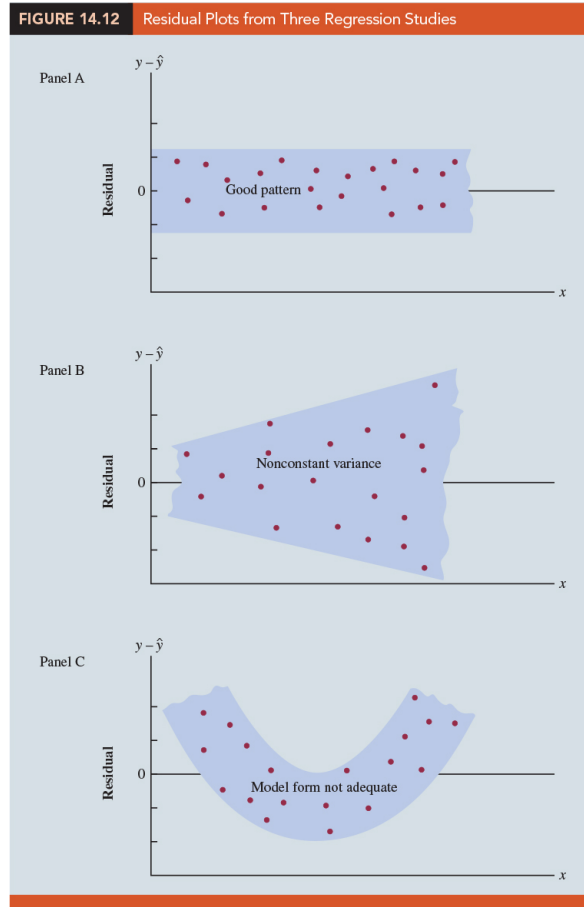
6. Much of residual analysis is based on an examination of graphical plots:

- (a) A plot of the residuals against values of the independent variable x .
- (b) A plot of residuals against the predicted values of the dependent variable y
- (c) A standardized residual plot.
- (d) A normal probability plot.

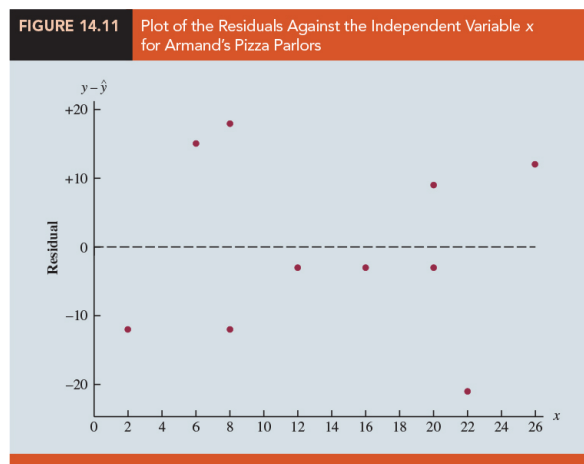
Residual Plot Against x

1. (Figure 14.12)

- (a) Panel A: If the assumption that the variance of ϵ is the same for all values of x and the assumed regression model is an adequate representation of the relationship between the variables, the residual plot should give an overall impression of a horizontal band of points.
- (b) Panel B: if the variance of ϵ is not the same for all values of x —for example, if variability about the regression line is greater for larger values of x .
- (c) Panel C: we would conclude that the assumed regression model is not an adequate representation of the relationship between the variables. A curvilinear regression model or multiple regression model should be considered.



2. (Figure 14.11) Example Armand's Pizza Parlors:

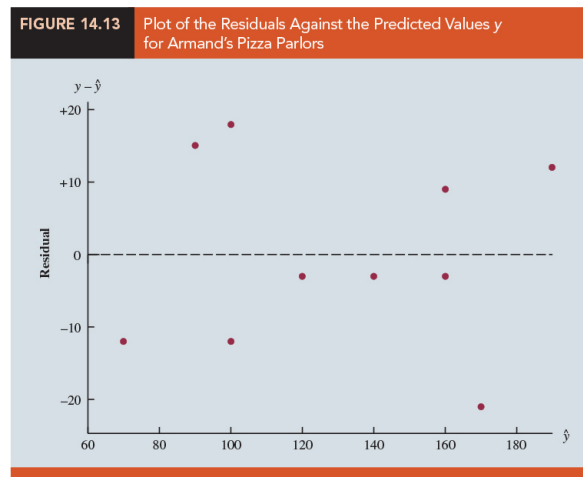


The residual plot does not provide evidence that the assumptions made for Armand's regression model should be challenged. At this point, we are confident in the conclusion that Armand's simple linear regression model is valid.

3. Experience and good judgment are always factors in the effective interpretation of residual plots.

Residual Plot Against \hat{y}

1. Another residual plot represents the predicted value of the dependent variable \hat{y} on the horizontal axis and the residual values on the vertical axis.
2. (Figure 14.13) With the Armand's data from Table 14.7,



Note that the pattern of this residual plot is the same as the pattern of the residual plot against the independent variable x .

3. For multiple regression analysis, the residual plot against \hat{y} is more widely used because of the presence of more than one independent variable.

Standardized Residuals

1. A random variable is standardized by subtracting its mean and dividing the result by its standard deviation.
2. With the least squares method, the mean of the residuals is zero. Thus, simply dividing each residual by its standard deviation provides the standardized residual.

3. Standard Deviation of the i th Residual

$$s_{y_i - \hat{y}_i} = \frac{s\sqrt{1 - h_i}}{\quad} \quad (14.30)$$

$s_{y_i - \hat{y}_i}$ = the standard deviation of residual i

s = the standard error of the estimate

$$h_i = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum(x_i - \bar{x})^2} \quad (14.31)$$

4. Standardized Residual for Observation i

$$\frac{y_i - \hat{y}_i}{s_{y_i - \hat{y}_i}} \quad (14.32)$$

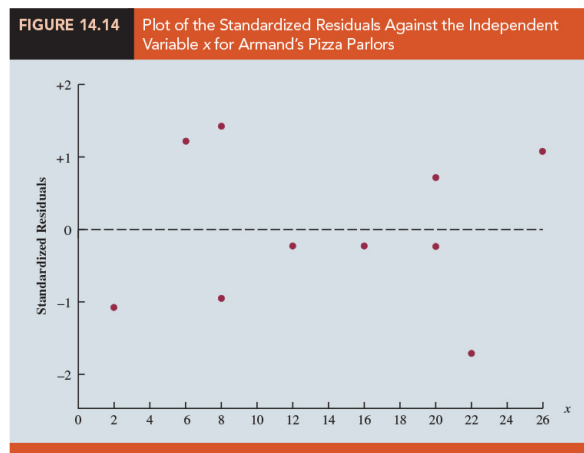
5. (Table 14.8) the standardized residuals for Armand's Pizza Parlors.

TABLE 14.8 Computation of Standardized Residuals for Armand's Pizza Parlors

Restaurant i	x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$\frac{(x_i - \bar{x})^2}{\sum(x_i - \bar{x})^2}$	h_i	$s_{y_i - \hat{y}_i}$	$y_i - \hat{y}_i$	Standardized Residual
1	2	-12	144	.2535	.3535	11.1193	-12	-1.0792
2	6	-8	64	.1127	.2127	12.2709	15	1.2224
3	8	-6	36	.0634	.1634	12.6493	-12	-.9487
4	8	-6	36	.0634	.1634	12.6493	18	1.4230
5	12	-2	4	.0070	.1070	13.0682	-3	-.2296
6	16	2	4	.0070	.1070	13.0682	-3	-.2296
7	20	6	36	.0634	.1634	12.6493	-3	-.2372
8	20	6	36	.0634	.1634	12.6493	9	.7115
9	22	8	64	.1127	.2127	12.2709	-21	-1.7114
10	26	12	144	.2535	.3535	11.1193	12	1.0792
		Total	568					

Note: The values of the residuals were computed in Table 14.7.

6. (Figure 14.14)



7. The standardized residual plot can provide insight about the assumption that the error term ϵ has a normal distribution. If this assumption is satisfied, the distribution of the standardized residuals should appear to come from a standard normal probability distribution.
8. Thus, when looking at a standardized residual plot, we should expect to see approximately 95% of the standardized residuals between -2 and +2.
9. We see in Figure 14.14 that for the Armand's example all standardized residuals are between -2 and $+2$. Therefore, on the basis of the standardized residuals, this plot gives us no reason to question the assumption that ϵ has a normal distribution.

Normal Probability Plot

1. Another approach for determining the validity of the assumption that the error term has a normal distribution is the normal probability plot.
2. To show how a normal probability plot is developed, we introduce the concept of normal scores.
 - (a) Suppose 10 values are selected randomly from a normal probability distribution with a mean of zero and a standard deviation of one, and that the sampling process is repeated over and over with the values in each sample of 10 ordered from smallest to largest.
 - (b) The random variable representing the smallest value obtained in repeated sampling is called the first-order statistic.
 - (c) Statisticians show that for samples of size 10 from a standard normal probability distribution, the expected value of the first-order statistic is -1.55 . This expected value is called a normal score.

NOTE: Compute the expected values of order statistics for a random sample from a standard normal distribution: `evNormOrdStats {EnvStats}`

<https://search.r-project.org/CRAN/refmans/EnvStats/html/evNormOrdStats.html>

- (d) (Table 14.9) For the case with a sample of size $n = 10$, there are 10 order statistics and 10 normal.

TABLE 14.9Normal Scores
For $n = 10$

Order Statistic	Normal Score
1	-1.55
2	-1.00
3	-.65
4	-.37
5	-.12
6	.12
7	.37
8	.65
9	1.00
10	1.55

TABLE 14.10Normal Scores and
Ordered Standardized
Residuals for Armand's
Pizza Parlors

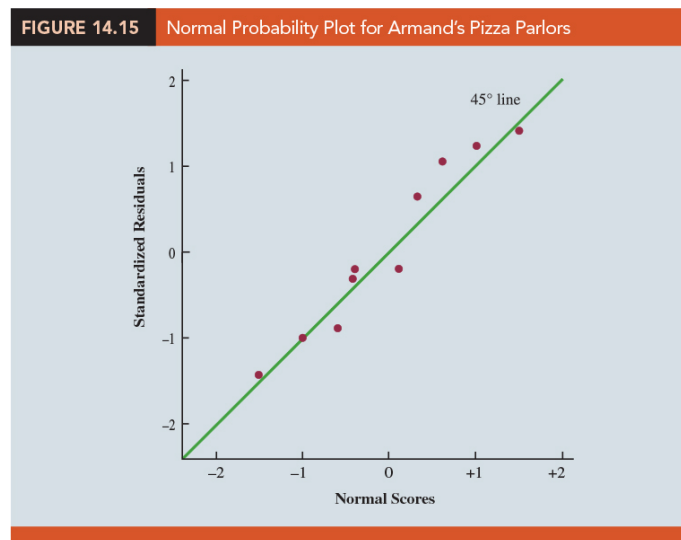
Normal Scores	Ordered Standardized Residuals
-1.55	-1.7114
-1.00	-1.0792
-.65	-.9487
-.37	-.2372
-.12	-.2296
.12	-.2296
.37	.7115
.65	1.0792
1.00	1.2224
1.55	1.4230

```
> data.frame(p, qnorm(p))
      p      qnorm.p
1 0.00000000      -Inf
2 0.09090909 -1.3351777
3 0.18181818 -0.9084579
4 0.27272727 -0.6045853
5 0.36363636 -0.3487557
6 0.45454545 -0.1141853
7 0.54545455  0.1141853
8 0.63636364  0.3487557
9 0.72727273  0.6045853
10 0.81818182  0.9084579
11 0.90909091  1.3351777
12 1.00000000      Inf
```

- (e) Let us now show how the 10 normal scores can be used to determine whether the standardized residuals for Armand's Pizza Parlors appear to come from a standard normal probability distribution.
- (f) (Table 14.10) The 10 normal scores and the ordered standardized residuals are shown together in Table 14.10. If the normality assumption is satisfied, the smallest standardized residual should be close to the smallest normal score, the next smallest standardized residual should be close to the next smallest

normal score, and so on.

- (g) A normal probability plot: a plot with the normal scores on the horizontal axis and the corresponding standardized residuals on the vertical axis.
- (h) If the standardized residuals are approximately normally distributed, the plotted points should cluster closely around a 45-degree line passing through the origin.
3. (Figure 14.15) the normal probability plot for the Armand's Pizza Parlors example: conclude that the assumption of the error term having a normal probability distribution is reasonable.



4. Any substantial curvature in the normal probability plot is evidence that the residuals have not come from a normal distribution.

14.9 Residual Analysis: Outliers and Influential Observations

Detecting Outliers

1. (Figure 14.16) is a scatter diagram for a data set that contains an outlier, a data point (observation) that does not fit the trend shown by the remaining data.

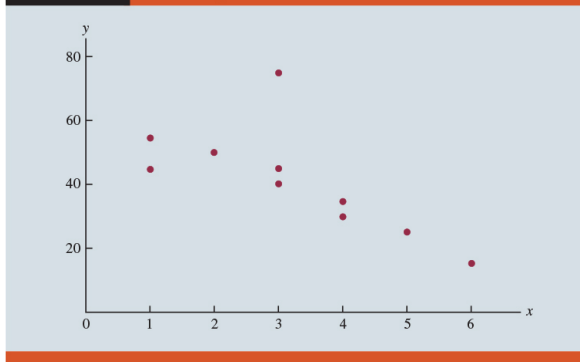


2. Outliers represent observations that are suspect and warrant careful examination. They may represent erroneous data; if so, the data should be corrected.
3. They may signal a violation of model assumptions; if so, another model should be considered.
4. Finally, they may simply be unusual values that occurred by chance. In this case, they should be retained.
5. (Table 14.11) The process of detecting outliers: Except for observation 4 ($x_4 = 3$, $y_4 = 75$), a pattern suggesting a negative linear relationship is apparent. Indeed, given the pattern of the rest of the data, we would expect y_4 to be much smaller and hence would identify the corresponding observation as an outlier.
6. For the case of simple linear regression, one can often detect outliers by simply examining the scatter diagram.
7. The standardized residuals can also be used to identify outliers. If an observation deviates greatly from the pattern of the rest of the data, the corresponding standardized residual will be large in absolute value.

TABLE 14.11
Data Set Illustrating the Effect of an Outlier

x_i	y_i
1	45
1	55
2	50
3	75
3	40
3	45
4	30
4	35
5	25
6	15

FIGURE 14.17 Scatter Diagram for Outlier Data Set



8. (Figure 14.18) the output from a regression analysis. The highlighted portion of the output shows that the standardized residual for observation 4 is 2.67. With normally distributed errors, standardized residuals should be outside the range of -2 to $+2$ approximately 5% of the time.

FIGURE 14.18 Output for Regression Analysis of the Outlier Data Set

Analysis of Variance					
Source	DF	Adj SS	Adj MS	F-Value	P-Value
Regression	1	1268.2	1268.2	7.90	.023
Error	8	1284.3	160.5		
Total	9	2552.5			

Model Summary		
S	R-sq	R-sq(adj)
12.6704	49.68%	43.39%

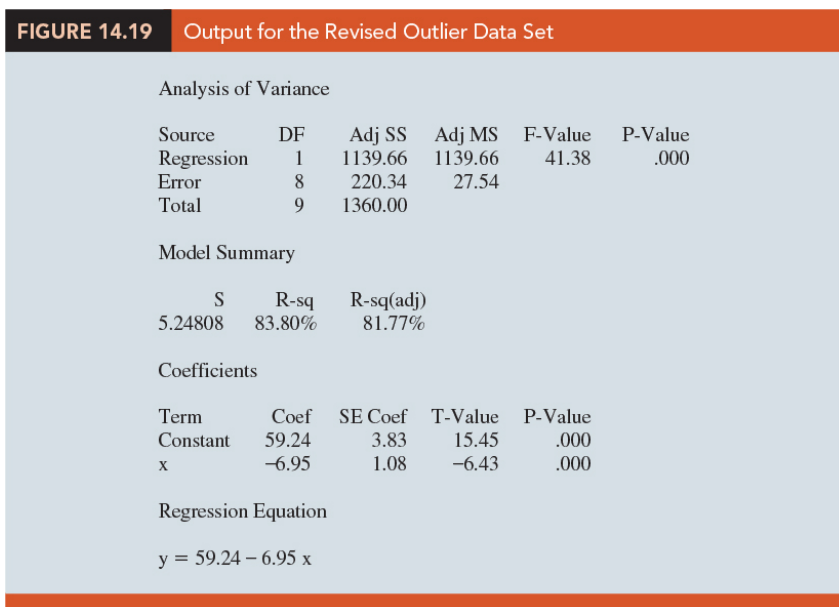
Coefficients				
Term	Coef	SE Coef	T-Value	P-Value
Constant	64.96	9.26	7.02	.000
x	-7.33	2.6	-2.81	.023

Regression Equation

$$y = 64.96 - 7.33 x$$

Observation	Predicted y	Residuals	Standard Residuals
1	57.6271	-12.6271	-1.0570
2	57.6271	-2.6271	-.2199
3	50.2966	-.2966	-.0248
4	42.9661	32.0339	2.6816
5	42.9661	-2.9661	-.2483
6	42.9661	2.0339	.1703
7	35.6356	-5.6356	-.4718
8	35.6356	-.6356	-.0532
9	28.3051	-3.3051	-.2767
10	20.9746	-5.9746	-.5001

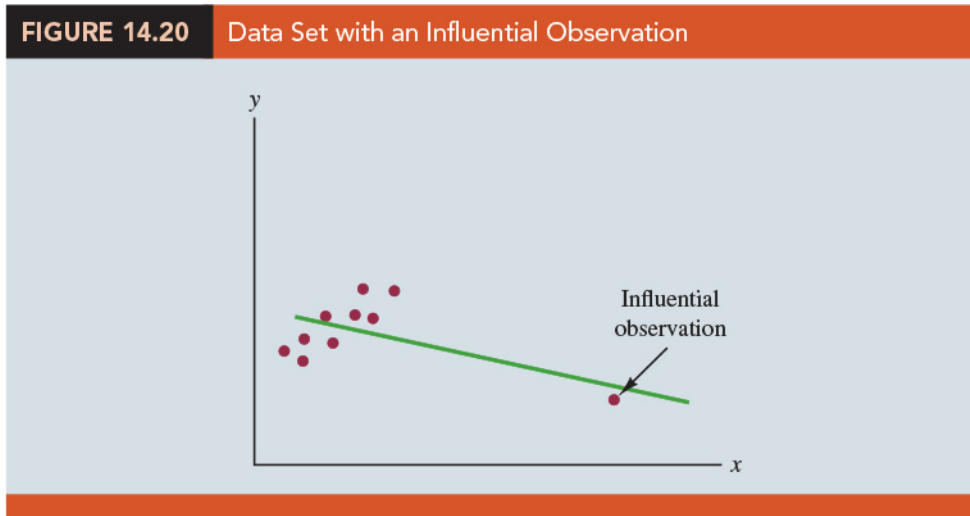
9. In deciding how to handle an outlier, we should first check to see whether it is a valid observation. Perhaps an error was made in initially recording the data or in entering the data into the computer file.
10. (Figure 14.19) For example, suppose that in checking the data for the outlier in Table 14.11, we find an error; the correct value for observation 4 is $x_4 = 3, y_4 = 30$. Figure 14.19 is a portion of the output obtained after correction of the value of y_4 . We see that using the incorrect data value substantially affected the goodness of fit. With the correct data, the value of R-sq increased from 49.68% to 83.8% and the value of b_0 decreased from 64.96 to 59.24. The slope of the line changed from -7.33 to -6.95 .



11. The identification of the outlier enabled us to correct the data error and improve the regression results.

Detecting Influential Observations

1. (Figure 14.20) shows an example of an influential observation in simple linear regression.



The estimated regression line has a negative slope. However, if the influential observation were dropped from the data set, the slope of the estimated regression line would change from negative to positive and the y -intercept would be smaller. Clearly, this one observation is much more influential in determining the estimated regression line than any of the others.

2. Influential observations can be identified from a scatter diagram when only one independent variable is present.
3. An influential observation may be an outlier (an observation with a y value that deviates substantially from the trend), it may correspond to an x value far away from its mean (e.g., see Figure 14.20), or it may be caused by a combination of the two (a somewhat off-trend y value and a somewhat extreme x value).
4. The presence of the influential observation in Figure 14.20, if valid, would suggest trying to obtain data on intermediate values of x to understand better the relationship between x and y .
5. Observations with extreme values for the independent variables are called high leverage points. The influential observation in Figure 14.20 is a point with high leverage.
6. The leverage of an observation is determined by how far the values of the independent variables are from their mean values.

7. For the single-independent-variable case, the leverage of the i th observation, denoted h_i , can be computed by using equation (14.33).

$$h_i = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2}$$

Definition and properties of leverages:

<https://online.stat.psu.edu/stat501/lesson/11/11.2>

8. From the formula, it is clear that the farther x_i is from its mean \bar{x} , the higher the leverage of observation i .
9. (Figure 14.21) a scatter diagram for the data set in Table 14.12, it is clear that observation 7 ($x = 70, y = 100$) is an observation with an extreme value of x . Hence, we would expect it to be identified as a point with high leverage:

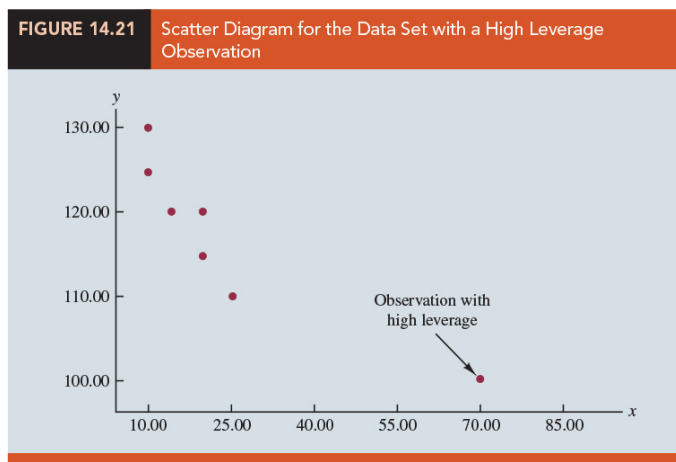
$$h_7 = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2} = \frac{1}{7} + \frac{(70 - 24.286)^2}{2621.43} = 0.94$$

10. For the case of simple linear regression, observations have high leverage if $h_i > 6/n$ or 0.99, whichever is smaller.
11. For the data set in Table 14.12, $6/n = 6/7 = 0.86$. Because $h_7 = 0.94 > 0.86$, we will identify observation 7 as an observation whose x value gives it large influence.
12. Influential observations that are caused by an interaction of large residuals and high leverage can be difficult to detect. Diagnostic procedures are available that take both into account in determining when an observation is influential. One such measure, called Cook's D statistic, will be discussed in Chapter 15.

TABLE 14.12

Data Set with a High Leverage Observation

x_i	y_i
10	125
10	130
15	120
20	115
20	120
25	110
70	100



☺ **EXERCISES**

14.2 : 1, 5, 6

14.3 : 15, 19, 20

14.5 : 23, 26, 27, 30

14.6 : 32, 36, 37

14.7 : 40, 41

14.8 : 45, 47

14.9 : 50, 52

SUP : 59, 67

“不要畏懼失敗，你應該要擔心沒有機會嘗試，但你有的是機會嘗試!”

“Don't fear failure. Be afraid of not having the chance, you have the chance!”

— 汽車總動員 3: 閃電再起 (*Cars 3*, 2017)

統計學 (二)

Anderson's Statistics for Business & Economics (14/E)

Chapter 15: Multiple Regression

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系級: _____ 學號: _____ 姓名: _____

15.1 Multiple Regression Model

1. (Recall) that the variable being predicted or explained is called the dependent variable and the variable being used to predict or explain the dependent variable is called the independent variable.
2. Multiple regression analysis is the study of how a dependent variable y is related to two or more independent variables. In the general case, we will use p to denote the number of independent variables.
3. The concepts of a regression model and a regression equation introduced in the preceding chapter are applicable in the multiple regression case.
4. **Multiple regression model:** The equation that describes how the dependent variable y is related to the independent variables x_1, x_2, \dots, x_p and an error term is called the multiple regression model.

$$\underline{y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p + \epsilon} \quad (15.1)$$

5. In the multiple regression model, $\beta_0, \beta_1, \beta_2, \dots, \beta_p$ are the parameters and the error term ϵ is a random variable. y is a linear function of x_1, x_2, \dots, x_p plus the error term ϵ .
6. The error term accounts for the variability in y that cannot be explained by the linear effect of the p independent variables.

7. **(Multiple regression equation):** The equation that describes how the mean value of y is related to x_1, x_2, \dots, x_p is called the multiple regression equation.

$$\underline{E(y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p} \quad (15.2)$$

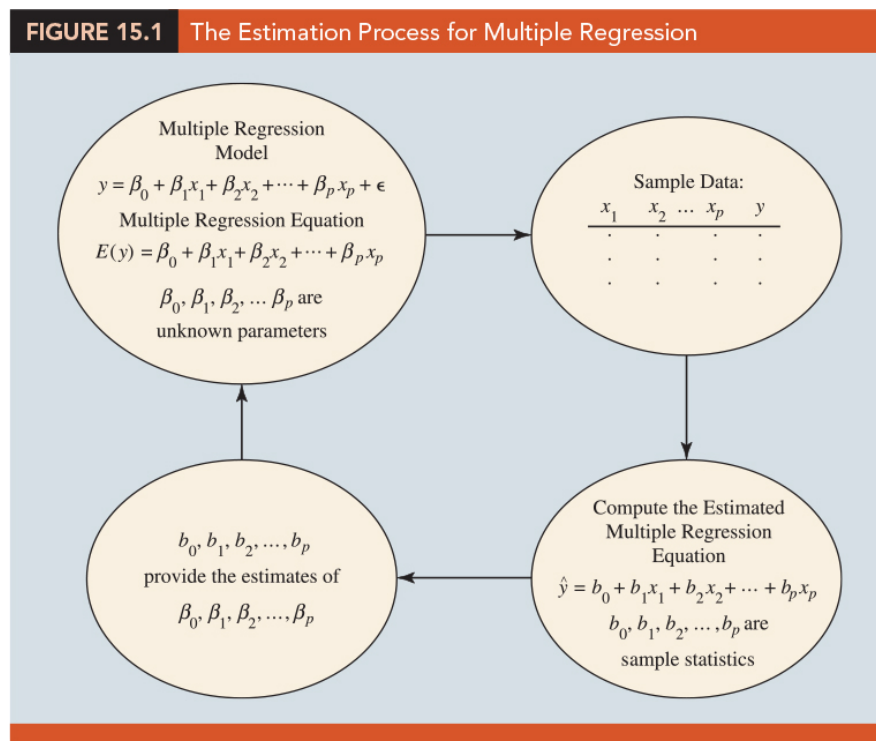
under the assumption that the mean or expected value of ϵ is zero.

8. **The estimated multiple regression equation:**

$$\underline{\hat{y} = b_0 + b_1 x_1 + b_2 x_2 + \dots + b_p x_p} \quad (15.3)$$

where $b_0, b_1, b_2, \dots, b_p$ are the estimates of $\beta_0, \beta_1, \beta_2, \dots, \beta_p$ and \hat{y} is the predicted value of the dependent variable

9. (Figure 15.1)



15.2 Least Squares Method

1. The least squares method is used to develop the estimated multiple regression equation:

$$\min \sum (y_i - \hat{y}_i)^2 \quad (15.4)$$

where y_i is observed value of the dependent variable for the i th observation, \hat{y}_i is predicted value of the dependent variable for the i th observation

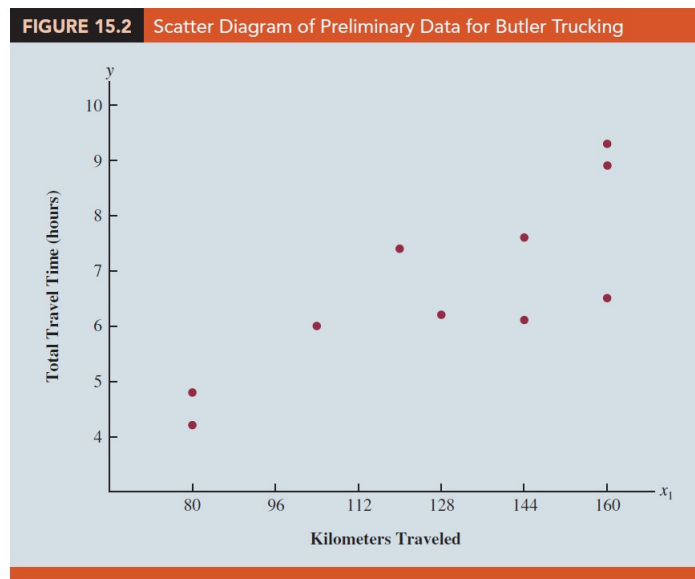
2. In multiple regression, however, the presentation of the formulas for the regression coefficients $\beta_0, \beta_1, \beta_2, \dots, \beta_p$ involves the use of matrix algebra and is beyond the scope of this text.
3. Therefore, in presenting multiple regression, we focus on how statistical software can be used to obtain the estimated regression equation and other information. The emphasis will be on how to interpret the computer output rather than on how to make the multiple regression computations.

An Example: Butler Trucking Company

1. The Butler Trucking Company, an independent trucking company in southern California.
2. A major portion of Butler's business involves deliveries throughout its local area. To develop better work schedules, the managers want to predict the total daily travel time for their drivers.
 - (a) Initially the managers believed that the total daily travel time would be closely related to the number of miles traveled in making the daily deliveries.
 - (b) (Table 15.1)(Figure 15.2) A simple random sample of 10 driving assignments provided the data shown in Table 15.1 and the scatter diagram shown in Figure 15.2.

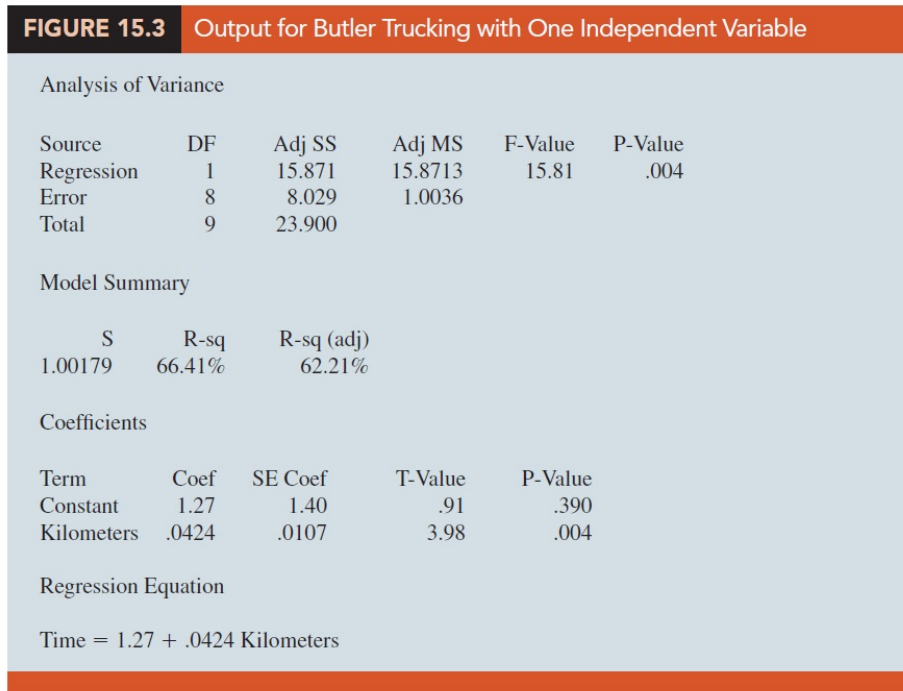
Driving Assignment	x_1 = Kilometers Traveled	y = Travel Time (hours)
1	160	9.3
2	80	4.8
3	160	8.9
4	160	6.5
5	80	4.2
6	128	6.2
7	120	7.4
8	104	6.0
9	144	7.6
10	144	6.1

Source: PC Magazine website, April, 2015. (<https://www.pcmag.com/reviews/monitors>)



- (c) After reviewing this scatter diagram, the managers hypothesized that the simple linear regression model $y = \beta_0 + \beta_1 x_1 + \epsilon$ could be used to describe the relationship between the total travel time (y) and the number of miles traveled (x_1).
- (d) (Figure 15.3) we show statistical software output from applying simple linear regression to the data in Table 15.1. The estimated regression equation is $\hat{y} = 1.27 + 0.0678x_1$
- At the 0.05 level of significance, the F value of 15.81 and its corresponding p -value of 0.004 indicate that the relationship is significant; that is, we can reject $H_0 : \beta_1 = 0$ because the p -value is less than $\alpha = 0.05$.
 - Note that the same conclusion is obtained from the t value of 3.98 and its associated p -value of 0.004.

- iii. Thus, we can conclude that the relationship between the total travel time and the number of miles traveled is significant ; longer travel times are associated with more miles traveled.



- iv. With a coefficient of determination (expressed as a percentage) of R-Sq = 66.41% , we see that 66.41% of the variability in travel time can be explained by the linear effect of the number of miles traveled.

3. (Table 15.2) The managers might want to consider adding a second independent variable (number of deliveries) to explain some of the remaining variability in the dependent variable.

TABLE 15.2 Data for Butler Trucking with Kilometers Traveled (x_1) and Number of Deliveries (x_2) as the Independent Variables

Driving Assignment	x_1 = Kilometers Traveled	x_2 = Number of Deliveries	y = Travel Time (hours)
1	160	4	9.3
2	80	3	4.8
3	160	4	8.9
4	160	2	6.5
5	80	2	4.2
6	128	2	6.2
7	120	3	7.4
8	104	4	6.0
9	144	3	7.6
10	144	2	6.1

4. (Figure 15.4) Computer output with both miles traveled (x_1) and number of deliveries (x_2) as independent variables is shown in Figure 15.4. The estimated regression equation is

$$\hat{y} = \underline{2.869 + .06113x_1 + .923x_2} \quad (15.6)$$

FIGURE 15.4 Output for Butler Trucking with Two Independent Variables

Analysis of Variance					
Source	DF	Adj SS	Adj MS	F-Value	P-Value
Regression	2	21.6006	10.8003	32.88	.000
Error	7	2.2994	.3285		
Total	9	23.900			

Model Summary		
S	R-sq	R-sq (adj)
.573142	90.38%	87.63%

Coefficients				
Term	Coef	SE Coef	T-Value	P-Value
Constant	-.869	.952	-.91	.392
Kilometers	.03821	.00618	6.18	.000
Deliveries	.923	.221	4.18	.004

Regression Equation
Time = -.869 + .03821 Kilometers + 0.923 Deliveries

Note on Interpretation of Coefficients

- One observation can be made at this point about the relationship between the estimated regression equation with only the miles traveled as an independent variable and the equation that includes the number of deliveries as a second independent variable.
- The value of β_1 is not the same in both cases. In simple linear regression, we interpret β_1 as an estimate of the change in y for a one-unit change in the independent variable.
- In multiple regression analysis, we interpret each regression coefficient as follows: b_i represents an estimate of the change in y corresponding to a one-unit change in x_i when all other independent variables are held constant.

4. Butler Trucking example

- (a) $\beta_1 = 0.06113$, an estimate of the expected increase in travel time corresponding to an increase of one mile in the distance traveled when the number of deliveries is held constant is 0.06113 hours.
- (b) $\beta_2 = 0.923$, an estimate of the expected increase in travel time corresponding to an increase of one delivery when the number of miles traveled is held constant is 0.923 hours.

15.3 Multiple Coefficient of Determination

1. In simple linear regression, we showed that the total sum of squares can be partitioned into two components: the sum of squares due to regression and the sum of squares due to error. The same procedure applies to the sum of squares in multiple regression.

$$\underline{SST = SSR + SSE} \quad (15.7)$$

where

$$\text{SST: total sum of squares} = \underline{\sum (y_i - \bar{y})^2}.$$

$$\text{SSR: sum of squares due to regression} = \underline{\sum (\hat{y}_i - \bar{y})^2}.$$

$$\text{SSE: sum of squares due to error} = \underline{\sum (y_i - \hat{y}_i)^2}.$$

2. **Example** Butler Trucking problem (Figure 15.4) $SST = 23.900$, $SSR = 21.6006$, and $SSE = 2.2994$.
3. With only one independent variable (number of miles traveled), the output in Figure 15.3 shows that $SST = 23.900$, $SSR = 15.871$, and $SSE = 8.029$. The value of SST is the same in both cases because it does not depend on \hat{y} , but SSR increases and SSE decreases when a second independent variable (number of deliveries) is added.

4. The multiple coefficient of determination, denoted R^2 , measures the goodness of fit for the estimated multiple regression equation.

$$R^2 = \frac{SSR}{SST} \quad (15.8)$$

5. The multiple coefficient of determination can be interpreted as the proportion of the variability in the dependent variable that can be explained by the estimated multiple regression equation.
6. Hence, when multiplied by 100, it can be interpreted as the percentage of the variability in y that can be explained by the estimated regression equation.
7. **Example** In the two-independent-variable Butler Trucking example, with $SSR = 21.6006$ and $SST = 23.900$, we have $R^2 = 21.6006/23.900 = 0.9038$.
8. Therefore, 90.38% of the variability in travel time y is explained by the estimated multiple regression equation with miles traveled and number of deliveries as the independent variables.
9. (Figure 15.3) the R-sq value for the estimated regression equation with only one independent variable, number of miles traveled (x_1), is 66.41%. Thus, the percentage of the variability in travel times that is explained by the estimated regression equation increases from 66.41% to 90.38% when number of deliveries is added as a second independent variable.
10. In general, R^2 always increases as independent variables are added to the model.
11. Many analysts prefer adjusting R^2 for the number of independent variables to avoid overestimating the impact of adding an independent variable on the amount of variability explained by the estimated regression equation.
12. With n denoting the number of observations and p denoting the number of independent variables, the adjusted multiple coefficient of determination is computed as follows:

$$R_a^2 = 1 - (1 - R^2) \frac{n - 1}{n - p - 1} \quad (15.9)$$

13. **Example** With $n = 10$ and $p = 2$, we have

$$R^2 = 1 - (1 - 0.9038) \frac{10 - 1}{10 - 2 - 1}$$

14. Thus, after adjusting for the two independent variables, we have an adjusted multiple coefficient of determination of 0.8763. This value (expressed as a percentage) is provided in the output in Figure 15.4 as R-Sq(adj) = 87.63%.
15. If the value of R^2 is small and the model contains a large number of independent variables, the adjusted coefficient of determination can take a negative value; in such cases, statistical software usually sets the adjusted coefficient of determination to zero.

15.4 Model Assumptions

1. The multiple regression model:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + \epsilon \quad (15.10)$$

2. The assumptions about the error term ϵ in the multiple regression model:

- (1) The error term ϵ is a random variable with mean or expected value of zero; that is, $E(\epsilon) = 0$.

Implication: For given values of x_1, x_2, \dots, x_p , the expected, or average, value of y is given by

$$E(y) = \underline{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p} \quad (15.11)$$

Equation (15.11) is the multiple regression equation. $E(y)$ represents the average of all possible values of y that might occur for the given values of x_1, x_2, \dots, x_p .

- (2) The variance of ϵ is denoted by σ^2 and is the same for all values of the independent variables x_1, x_2, \dots, x_p ; that is, $Var(\epsilon) = \sigma^2$.

Implication: The variance of y about the regression line equals σ^2 and is the same for all values of x_1, x_2, \dots, x_p .

- (3) The values of ϵ are independent.

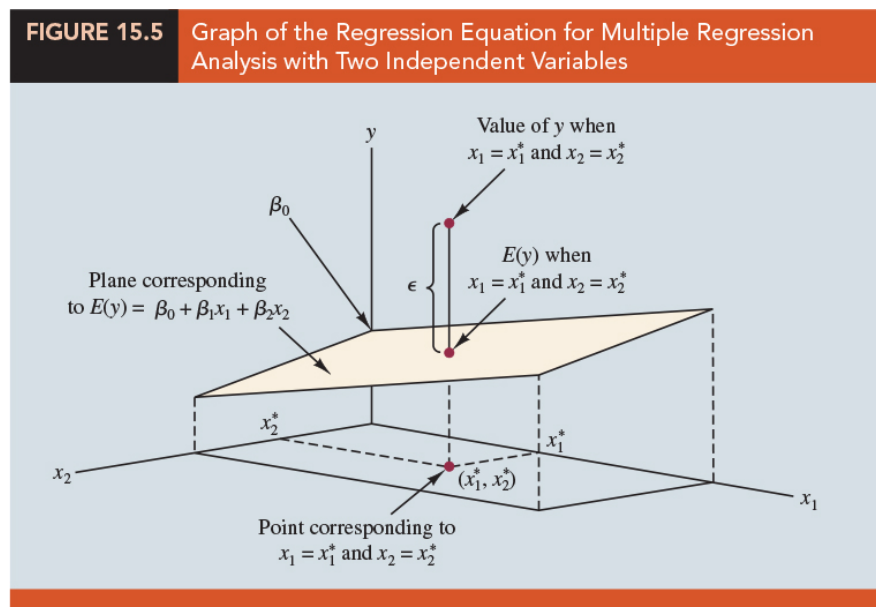
Implication: The value of ϵ for a particular set of values for the independent variables is not related to the value of ϵ for any other set of values.

- (4) The error term ϵ is a normally distributed random variable reflecting the deviation between the y value and the $E(y)$ given by $\beta_0 + \beta_1x_1 + \beta_2x_2 + \cdots + \beta_px_p$.

Implication: Because $\beta_0, \beta_1, \dots, \beta_p$ are constants for the given values of x_1, x_2, \dots, x_p , the dependent variable y is also a normally distributed random variable.

3. (Figure 15.5) Consider the following two-independent-variable multiple regression equation.

$$E(y) = \beta_0 + \beta_1x_1 + \beta_2x_2$$



4. Note that the value of ϵ shown is the difference between the actual y value and the expected value of y , $E(y)$, when $x_1 = x_1^*$ and $x_2 = x_2^*$.
5. In regression analysis, the term response variable is often used in place of the term dependent variable. Furthermore, since the multiple regression equation generates a plane or surface, its graph is called a response surface.

15.5 Testing for Significance

- In simple linear regression, both t test and an F test provide the same conclusion; that is, if the null hypothesis is rejected, we conclude that $\beta_1 \neq 0$.
- In multiple regression, the t test and the F test have different purposes.
 - The F test is used to determine whether a significant relationship exists between the dependent variable and the set of all the independent variables; we will refer to the F test as the test for overall significance.
 - If the F test shows an overall significance, the t test is used to determine whether each of the individual independent variables is significant. A separate t test is conducted for each of the independent variables in the model; we refer to each of these t tests as a test for individual significance.
- In the material that follows, we will explain the F test and the t test and apply each to the Butler Trucking Company example.

F Test

- The hypotheses for the F test involve the parameters of the multiple regression model.

$$H_0 : \underline{\beta_1 = \beta_2 = \cdots = \beta_p = 0}$$

H_a : One or more of the parameters are not equal to zero

- If H_0 is rejected, the test gives us sufficient statistical evidence to conclude that one or more of the parameters are not equal to zero and that the overall relationship between y and the set of independent variables x_1, x_2, \dots, x_p is significant.
- However, if H_0 cannot be rejected, we do not have sufficient evidence to conclude that a significant relationship is present.
- (Review)(Chapter 14)
 - A mean square is a sum of squares divided by its corresponding degrees of freedom.

- (b) In the multiple regression case, the total sum of squares (SST) has $\underline{n-1}$ degrees of freedom, the sum of squares due to regression (SSR) has \underline{p} degrees of freedom, and the sum of squares due to error (SSE) has $\underline{n-p-1}$ degrees of freedom.
- (c) Hence, the mean square due to regression (MSR) is $\underline{SSR/p}$ and the mean square due to error (MSE) is $\underline{SSE/(n-p-1)}$.
- (d) MSE provides an unbiased estimate of $\underline{\sigma^2}$, the variance of the error term ϵ .
- (e) If $\underline{H_0 : \beta_1 = \beta_2 = \dots = \beta_p = 0}$ is true, \underline{MSR} also provides an unbiased estimate of σ^2 , and the value of MSR/MSE should be close to $\underline{1}$.
- (f) However, if H_0 is false, MSR overestimates σ^2 and the value of MSR/MSE becomes larger.
5. To determine how large the value of $\underline{MSR/MSE}$ must be to reject H_0 , we make use of the fact that if $\underline{H_0}$ is true and the assumptions about the multiple regression model are valid, the sampling distribution of MSR/MSE is an \underline{F} distribution with \underline{p} degrees of freedom in the numerator and $\underline{n-p-1}$ in the denominator.

6. F test for overall significance

(a) Hypothesis:

$$H_0 : \beta_1 = \beta_2 = \dots = \beta_p = 0$$

H_a : One or more of the parameters are not equal to zero

(b) Test statistic:

$$\underline{F = \frac{MSR}{MSE}} \quad (15.14)$$

(c) Rejection rule:

i. p -value approach: Reject H_0 if $\underline{p\text{-value} \leq \alpha}$.

ii. Critical value approach: Reject H_0 if $\underline{F \geq F_{\alpha, p, n-p-1}}$

TABLE 15.3 ANOVA Table for a Multiple Regression Model with p Independent Variables

Source	Sum of Squares	Degrees of Freedom	Mean Square	F
Regression	SSR	p	$MSR = \frac{SSR}{p}$	$F = \frac{MSR}{MSE}$
Error	SSE	$n - p - 1$	$MSE = \frac{SSE}{n - p - 1}$	
Total	SST	$n - 1$		

7. **Example** Butler Trucking Company

(a) Hypotheses:

$$H_0 : \beta_1 = \beta_2 = 0$$

$$H_a : \beta_1 \text{ and/or } \beta_2 \text{ is not equal to zero}$$

(b) (Figure 15.6)

FIGURE 15.6 Output for Butler Trucking with Two Independent Variables, Kilometers Traveled (x_1) and Number of Deliveries (x_2)

Analysis of Variance					
Source	DF	Adj SS	Adj MS	F-Value	P-Value
Regression	2	21.6006	10.8003	32.88	.000
Error	7	2.2994	.3285		
Total	9	23.900			

Model Summary		
S	R-sq	R-sq (adj)
.573142	90.38%	87.63%

Coefficients				
Term	Coef	SE Coef	T-Value	P-Value
Constant	-.869	.952	-.91	.392
Kilometers	.03821	.00618	6.18	.000
Deliveries	.923	.221	4.18	.004

Regression Equation

Time = $-.869 + .03821 \text{ Kilometers} + .923 \text{ Deliveries}$

(c) $MSR = 10.8003$ and $MSE = 0.3285$, $F = \underline{32.88}$. Using $\alpha = 0.01$, $\underline{F_{0.01,2,7} = 9.55}$. With $F = 32.88 > 9.55$, we reject $H_0 : \beta_1 = \beta_2 = 0$.

- (d) Using $\alpha = 0.01$, the p -value = 0.000 indicates that we can reject $H_0 : \beta_1 = \beta_2 = 0$ because the p -value is less than $\alpha = 0.01$.
- (e) Conclude that a significant relationship is present between travel time y and the two independent variables, miles traveled and number of deliveries.

t Test

1. If the F test shows that the multiple regression relationship is significant, a t test can be conducted to determine the significance of each of the individual parameters.

2. The t test for individual significance

- (a) Hypothesis: For any parameter β_i

$$H_0 : \underline{\beta_i = 0}$$

$$H_a : \beta_i \neq 0$$

- (b) Test statistic:

$$t = \frac{b_i}{s_{b_i}} \quad (15.15)$$

- (c) Rejection rule:

i. p -value approach: Reject H_0 if p -value $\leq \alpha$.

ii. Critical value approach: Reject H_0 if $t \geq t_{\alpha/2, n-p-1}$ or if $t \leq -t_{\alpha/2, n-p-1}$.

3. In the test statistic, s_{b_i} is the estimate of the standard deviation of b_i . The value of s_{b_i} will be provided by the computer software package.

補充:

The multiple regression model

$$\underline{y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_{p-1} x_{p-1} + \epsilon}$$

or

$$\underline{y_i = \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_{p-1} x_{i,p-1} + \epsilon_i, \quad i = 1, \dots, n}$$

- (a) In the matrix notation:

$$\underline{\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}} \quad \text{or} \quad \mathbf{y}_{n \times 1} = \mathbf{X}_{n \times p} \boldsymbol{\beta}_{p \times 1} + \boldsymbol{\epsilon}_{n \times 1}$$

(b) Design matrix \mathbf{X} :

$$\mathbf{X} =$$

(c) Use Least-squares to fit a regression line to the data $\{\mathbf{x}_i, y_i\}_{i=1}^n$, where $\mathbf{x}_i = \{x_{i,1}, x_{i,2}, \dots, x_{i,p-1}\}$

$$\begin{aligned} Q(\boldsymbol{\beta}) &= (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) = \sum_{i=1}^n (y_i - \mathbf{x}_i\boldsymbol{\beta})^2. \\ \frac{\partial Q}{\partial \boldsymbol{\beta}} &= -2\mathbf{X}'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) = \mathbf{0} \\ \Rightarrow &(\mathbf{X}'\mathbf{X})\boldsymbol{\beta} = \mathbf{X}'\mathbf{y} \\ \Rightarrow &\hat{\boldsymbol{\beta}} = \mathbf{b} = \underline{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}} \end{aligned}$$

(d) Variance of the sampling distribution of $b_i, i = 1, 2, \dots, p$.

$$Var(b_i) = \frac{\sigma^2}{(n-1)S_{x_i}^2(1-R_i^2)},$$

where $S_{x_i}^2$ is the sample variance of variable x_i and R_i^2 is R -square of the regression of x_i on the rest of the explanatory variables of the models (including the constant term). Note that the variance should be conditional on the observed values of the explanatory variables.

4. Example Butler Trucking Company

(a) (Figure 15.6) that shows the output for the t -ratio calculations:

$$b_1 = 0.06113, b_2 = 0.923, s_{b_1} = 0.00989, s_{b_2} = 0.221$$

(b) The test statistic for the hypotheses involving parameters β_1 and β_2 :

$$t = 0.06113/0.00989 = 6.18, \quad t = 0.923/0.221 = 4.18$$

(c) Using $\alpha = 0.01$, the p -values of 0.000 and 0.004 in the output indicate that we can reject $H_0 : \beta_1 = 0$ and $H_0 : \beta_2 = 0$. Hence, both parameters are statistically significant.

(d) Alternatively, $t_{0.005,7} = 3.499$. With $6.18 > 3.499$, we reject $H_0 : \beta_1 = 0$. Similarly, with $4.18 > 3.499$, we reject $H_0 : \beta_2 = 0$.

Multicollinearity

1. We use the term independent variable in regression analysis to refer to any variable being used to predict or explain the value of the dependent variable.
2. The term does not mean, however, that the independent variables themselves are independent in any statistical sense. On the contrary, most independent variables in a multiple regression problem are correlated to some degree with one another.
3. Example Butler Trucking Example
 - (a) Butler Trucking example involves the two independent variables x_1 (miles traveled) and x_2 (number of deliveries), we could treat the miles traveled as the dependent variable and the number of deliveries as the independent variable to determine whether those two variables are themselves related.
 - (b) Compute the sample correlation coefficient $r(x_1, x_2) = 0.16$ and find that some degree of linear association between the two independent variables.
4. In multiple regression analysis, multicollinearity refers to the correlation among the independent variables.
5. Example Modified Butler Trucking Example, the potential problems of multicollinearity.
 - (a) Consider a modification of the Butler Trucking example. Instead of x_2 being the number of deliveries, let x_2 denote the number of gallons of gasoline consumed. Clearly, x_1 (the miles traveled) and x_2 are related; that is, we know that the number of gallons of gasoline used depends on the number of miles traveled.
 - (b) We would conclude logically that x_1 and x_2 are highly correlated independent variables.
 - (c) Assume that we obtain the equation $\hat{y} = b_0 + b_1x_1 + b_2x_2$ and find that the F test shows the relationship to be significant. Then suppose we conduct a t test on β_1 to determine whether $\beta_1 \neq 0$, and we cannot reject $H_0 : \beta_1 = 0$. Does this result mean that travel time is not related to miles traveled? Not necessarily.

- (d) What it probably means is that with x_2 already in the model, x_1 does not make a significant contribution to determining the value of y .
- (e) This interpretation makes sense in our example; if we know the amount of gasoline consumed (x_2), we do not gain much additional information useful in predicting y by knowing the miles traveled (x_1).
- (f) Similarly, a t test might lead us to conclude $\beta_2 = 0$ on the grounds that, with x_1 in the model, knowledge of the amount of gasoline consumed does not add much.
6. To summarize, in t tests for the significance of individual parameters, the difficulty caused by multicollinearity is that it is possible to conclude that none of the individual parameters is significantly different from zero when an F test on the overall multiple regression equation indicates a significant relationship.
7. Statisticians have developed several tests for determining whether multicollinearity is high enough to cause problems.
8. According to the rule of thumb test, multicollinearity is a potential problem if the absolute value of the sample correlation coefficient exceeds 0.7 for any two of the independent variables.
9. The other types of tests are more advanced and beyond the scope of this text. If possible, every attempt should be made to avoid including independent variables that are highly correlated.
10. When multicollinearity is severe,
- (a) it is not possible to determine the separate effect of any particular independent variable on the dependent variable.
 - (b) we can have difficulty interpreting the results of t tests on the individual parameters.
 - (c) Least squares estimates may have the wrong sign.
11. 補充:
- (a) Multicollinearity in Regression Analysis: Problems, Detection, and Solutions

<https://statisticsbyjim.com/regression/multicollinearity-in-regression-analysis/>

- (b) Multicollinearity in Regression: Why it is a problem? How to check and fix it

<https://towardsdatascience.com/multi-collinearity-in-regression-fe7a2c1467ea>

- (c) Eight Ways to Detect Multicollinearity

<https://www.theanalysisfactor.com/eight-ways-to-detect-multicollinearity/>

- (d) Multicollinearity (Wikipedia)

<https://en.wikipedia.org/wiki/Multicollinearity>

15.6 Using the Estimated Regression Equation for Estimation and Prediction

1. The procedures for estimating the mean value of y and predicting an individual value of y in multiple regression are similar to those in regression analysis involving one independent variable.
2. We substitute the given values of x_1, x_2, \dots, x_p into the estimated regression equation and use the corresponding value of \hat{y} as the point estimate.
3. **Example** Butler Trucking example
 - (a) We want to use the estimated regression equation involving x_1 (miles traveled) and x_2 (number of deliveries) to develop two interval estimates:
 - i. A confidence interval of the mean travel time for all trucks that travel 100 miles and make two deliveries.
 - ii. A prediction interval of the travel time for one specific truck that travels 100 miles and makes two deliveries
 - (b) Using the estimated regression equation $\hat{y} = -0.869 + 0.06113x_1 + 0.923x_2$ with $x_1 = 100$ and $x_2 = 2$, we obtain

$$\hat{y} = \underline{2.869 + 0.06113(100) + 0.923(2) = 7.09}$$

Hence, the point estimate of travel time in both cases is approximately seven hours.

- (c) To develop interval estimates for the mean value of y and for an individual value of y , we use a procedure similar to that for regression analysis involving one independent variable. The formulas required are beyond the scope of the text, but statistical software for multiple regression analysis will often provide confidence intervals once the values of x_1, x_2, \dots, x_p are specified by the user.
- (d) (Table 15.4)

Value of x_1	Value of x_2	95% Confidence Interval		95% Prediction Interval	
		Lower Limit	Upper Limit	Lower Limit	Upper Limit
160	4	8.135	9.742	7.363	10.514
80	3	4.127	5.789	3.369	6.548
160	4	8.135	9.742	7.363	10.514
160	2	6.258	7.925	5.500	8.683
80	2	3.146	4.924	2.414	5.656
128	2	5.232	6.505	4.372	7.366
120	3	6.037	6.936	5.059	7.915
104	4	5.960	7.637	5.205	8.392
144	3	6.917	7.891	5.964	8.844
144	2	5.776	7.184	4.953	8.007
120	4	6.669	8.152	5.865	8.955

- (e) Note that the interval estimate for an individual value of y is wider than the interval estimate for the expected value of y . This difference simply reflects the fact that for given values of x_1 and x_2 we can estimate the mean travel time for all trucks with more precision than we can predict the travel time for one specific truck.

15.7 Categorical Independent Variables

- (a) Thus far, the examples we have considered involved quantitative independent variables such as student population, distance traveled, and number of deliveries.
- (b) In many situations, however, we must work with categorical independent variables such as gender (male, female), method of payment (cash, credit card, check), and so on.

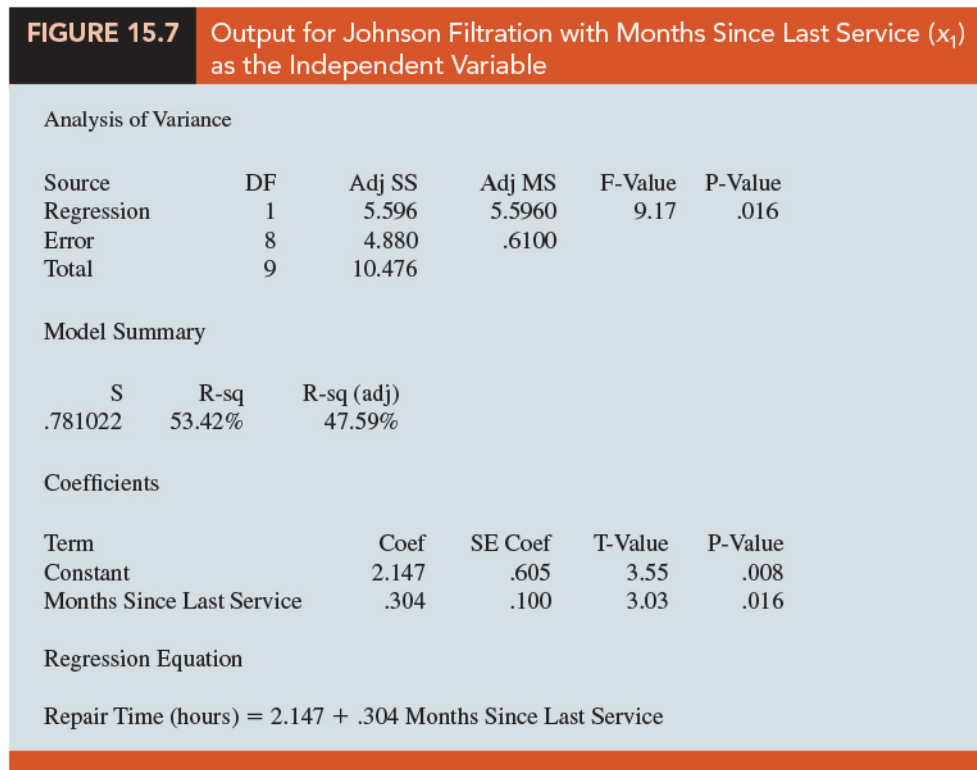
An Example: Johnson Filtration, Inc.

- (a) (Background) Johnson Filtration, Inc., provides maintenance service for water-filtration systems throughout southern Florida. Customers contact Johnson with requests for maintenance service on their water-filtration systems. To estimate the service time and the service cost, Johnson's managers want to predict the repair time necessary for each maintenance request.
- (b) (Dependent variable/Independent variables) Hence, repair time in hours is the dependent variable. Repair time is believed to be related to two factors, the number of months since the last maintenance service and the type of repair problem (mechanical or electrical).
- (c) (Data)(Table 15.5)

Service Call	Months Since Last Service	Type of Repair	Repair Time in Hours
1	2	Electrical	2.9
2	6	Mechanical	3.0
3	8	Electrical	4.8
4	3	Mechanical	1.8
5	2	Electrical	2.9
6	7	Electrical	4.9
7	9	Mechanical	4.2
8	8	Mechanical	4.8
9	4	Electrical	4.4
10	6	Electrical	4.5

- (d) (SLR) Let y denote the repair time in hours and x_1 denote the number of months since the last maintenance service. The regression model that uses only x_1 to predict y is $y = \beta_0 + \beta_1 x_1 + \epsilon$

(e) (Figure 15.7)



- i. The estimated regression equation is $\hat{y} = 2.147 + 0.304x_1$.
- ii. At the 0.05 level of significance, the p -value of 0.016 for the t (or F) test indicates that the number of months since the last service is significantly related to repair time.
- iii. R -sq = 53.42% indicates that x_1 alone explains 53.42% of the variability in repair time.

4. To incorporate the type of repair into the regression model, we define

$$x_2 = \begin{cases} \underline{0}, & \text{if the type of repair is mechanical} \\ \underline{1}, & \text{if the type of repair is electrical} \end{cases}$$

5. In regression analysis x_2 is called a dummy or indicator variable.
6. Using this dummy variable, we can write the multiple regression model as

$$y = \underline{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon}$$

7. (Table 15.6) Data for the Johnson Filtration Example with Type of Repair Indicated by a Dummy Variable ($x_2 = 0$ for Mechanical; $x_2 = 1$ for Electrical)

Customer	Months Since Last Service (x_1)	Type of Repair (x_2)	Repair Time in Hours (y)
1	2	1	2.9
2	6	0	3.0
3	8	1	4.8
4	3	0	1.8
5	2	1	2.9
6	7	1	4.9
7	9	0	4.2
8	8	0	4.8
9	4	1	4.4
10	6	1	4.5

8. (Figure 15.7) Output for Johnson Filtration with Months Since Last Service (x_1) as the Independent Variable

Analysis of Variance					
Source	DF	Adj SS	Adj MS	F-Value	P-Value
Regression	2	9.0009	4.50046	21.36	.001
Error	7	1.4751	.21073		
Total	9	10.4760			
Model Summary					
S	R-sq	R-sq (adj)			
.459048	85.92%	81.90%			
Coefficients					
Term	Coef	SE Coef	T-Value	P-Value	
Constant	.930	.467	1.99	.087	
Months Since Last Service	.3876	.0626	6.20	.000	
Type of Repair	1.263	.314	4.02	.005	
Regression Equation					
Repair Time (hours) = .930 + .3876 Months Since Last Service + 1.263 Type of Repair					

- (a) The estimated multiple regression equation is

$$\hat{y} = 0.93 + 0.3876x_1 + 1.263x_2 \quad (15.17)$$

- (b) At the 0.05 level of significance, the p -value of 0.001 associated with the F test ($F = 21.36$) indicates that the regression relationship is significant.

- (c) The t test shows that both months since last service (p -value = 0.000) and type of repair (p -value = 0.005) are statistically significant.
- (d) In addition, R -Sq = 85.92% and R -Sq (adj) = 81.9% indicate that the estimated regression equation does a good job of explaining the variability in repair times.
- (e) Thus, equation (15.17) should prove helpful in predicting the repair time necessary for the various service calls.

Interpreting the Parameters

1. The multiple regression equation for the Johnson Filtration example is

$$E(y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \quad (15.18)$$

2. Consider the case when $x_2 = 0$ (mechanical repair). Using $E(y|\text{mechanical})$ to denote the mean or expected value of repair time given a mechanical repair, we have

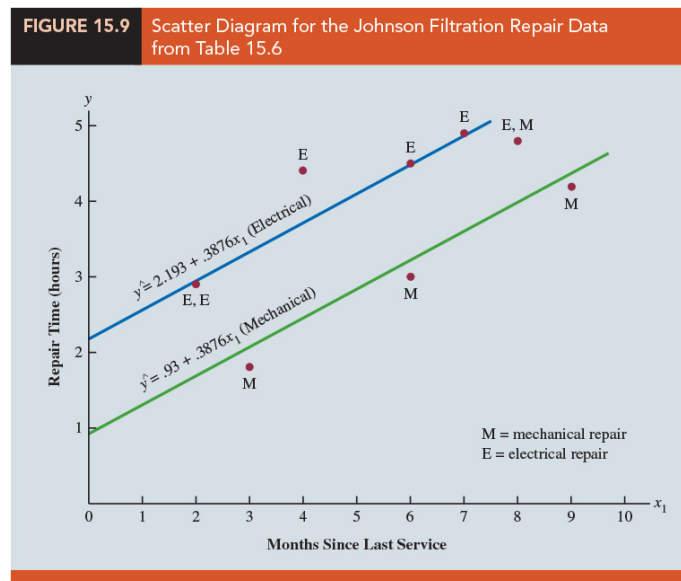
$$E(y|\text{mechanical}) = \underline{\beta_0 + \beta_1 x_1 + \beta_2(0)} = \underline{\beta_0 + \beta_1 x_1} \quad (15.19)$$

3. Similarly, for an electrical repair ($x_2 = 1$), we have

$$E(y|\text{electrical}) = \underline{\beta_0 + \beta_1 x_1 + \beta_2(1)} = \underline{(\beta_0 + \beta_2) + \beta_1 x_1} \quad (15.20)$$

4. Comparing equations (15.19) and (15.20), we see that the mean repair time is a linear function of x_1 for both mechanical and electrical repairs. The slope of both equations is β_1 , but the y -intercept differs.
5. The y -intercept is β_0 in equation (15.19) for mechanical repairs and $(\beta_0 + \beta_2)$ in equation (15.20) for electrical repairs.
6. The interpretation of β_2 is that it indicates the difference between the mean repair time for an electrical repair and the mean repair time for a mechanical repair.
- (a) If $\beta_2 > 0$, the mean repair time for an electrical repair will be greater than that for a mechanical repair;

- (b) if $\beta_2 < 0$, the mean repair time for an electrical repair will be less than that for a mechanical repair.
- (c) if $\beta_2 = 0$, there is no difference in the mean repair time between electrical and mechanical repairs and the type of repair is not related to the repair time.
7. Using the estimated multiple regression equation $\hat{y} = 0.93 + 0.3876x_1 + 1.263x_2$, we see that 0.93 is the estimate of β_0 and 1.263 is the estimate of β_2 .
8. Thus, when $x_2 = 0$ (mechanical repair)
- $$\hat{y} = 0.93 + 0.3876x_1 \quad (15.21)$$
- and when $x_2 = 1$ (electrical repair)
- $$\hat{y} = 0.93 + 0.3876x_1 + 1.263(1) = 2.193 + 0.3876x_1 \quad (15.22)$$
9. In effect, the use of a dummy variable for type of repair provides two estimated regression equations that can be used to predict the repair time, one corresponding to mechanical repairs and one corresponding to electrical repairs.
10. In addition, with $\beta_2 = 1.263$, we learn that, on average, electrical repairs require 1.263 hours longer than mechanical repairs.
11. (Figure 15.9) Scatter Diagram for the Johnson Filtration Repair Data



More Complex Categorical Variables

1. If a categorical variable has k levels, $k-1$ dummy variables are required, with each dummy variable being coded as 0 or 1.
2. **Example** Suppose a manufacturer of copy machines organized the sales territories for a particular state into three regions: A, B, and C. The managers want to use regression analysis to help predict the number of copiers sold per week.
3. With the number of units sold as the dependent variable, they are considering several independent variables (the number of sales personnel, advertising expenditures, and so on).
4. Suppose the managers believe sales region is also an important factor in predicting the number of copiers sold. Because sales region is a categorical variable with three levels, A, B and C, we will need $3-1=2$ dummy variables to represent the sales region. Each variable can be coded 0 or 1:

$$x_1 = \begin{cases} 1, & \text{if sales region B} \\ 0, & \text{otherwise} \end{cases}$$

$$x_2 = \begin{cases} 1, & \text{if sales region C} \\ 0, & \text{otherwise} \end{cases}$$

5. We have the following values of x_1 and x_2 :

Region	x_1	x_2
A	0	0
B	1	0
C	0	1

6. Observations corresponding to region A would be coded $x_1 = 0, x_2 = 0$; observations corresponding to region B would be coded $x_1 = 1, x_2 = 0$; and observations corresponding to region C would be coded $x_1 = 0, x_2 = 1$.
7. The regression equation relating the expected value of the number of units sold, $E(y)$, to the dummy variables would be written as

$$E(y) = \underline{\beta_0 + \beta_1 x_1 + \beta_2 x_2}$$

8. To help us interpret the parameters β_0 , β_1 , and β_2 , consider the following three variations of the regression equation.

$$E(y|\text{region A}) = \beta_0 + \beta_1(0) + \beta_2(0) = \beta_0$$

$$E(y|\text{region B}) = \beta_0 + \beta_1(1) + \beta_2(0) = \beta_0 + \beta_1$$

$$E(y|\text{region C}) = \beta_0 + \beta_1(0) + \beta_2(1) = \beta_0 + \beta_2$$

- (a) Thus, β_0 is the mean or expected value of sales for region A ;
- (b) β_1 is the difference between the mean number of units sold in region B and the mean number of units sold in region A ;
- (c) and β_2 is the difference between the mean number of units sold in region C and the mean number of units sold in region A .
9. Two dummy variables were required because sales region is a categorical variable with three levels.
10. The assignment was arbitrary . For example, we could have chosen $x_1 = 1, x_2 = 0$ to indicate region A, $x_1 = 0, x_2 = 0$ to indicate region B, and $x_1 = 0, x_2 = 1$ to indicate region C.

Region	x_1	x_2
A	1	0
B	0	0
C	0	1

In that case, β_1 would have been interpreted as the mean difference between regions A and B and β_2 as the mean difference between regions C and B.

11. The important point to remember is that when a categorical variable has k levels, $k-1$ dummy variables are required in the multiple regression analysis. Thus, if the sales region example had a fourth region, labeled D, three dummy variables would be necessary. For example, the three dummy variables can be coded as follows.

$$x_1 = \begin{cases} 1, & \text{if sales region B} \\ 0, & \text{otherwise} \end{cases} \quad x_2 = \begin{cases} 1, & \text{if sales region C} \\ 0, & \text{otherwise} \end{cases} \quad x_3 = \begin{cases} 1, & \text{if sales region D} \\ 0, & \text{otherwise} \end{cases}$$

15.8 Residual Analysis

1. Standardized Residual for Observation i

$$\frac{y_i - \hat{y}_i}{s_{y_i - \hat{y}_i}} \quad (15.23)$$

where $s_{y_i - \hat{y}_i}$ is the standard deviation of residual i .

2. Standard Deviation of Residual i

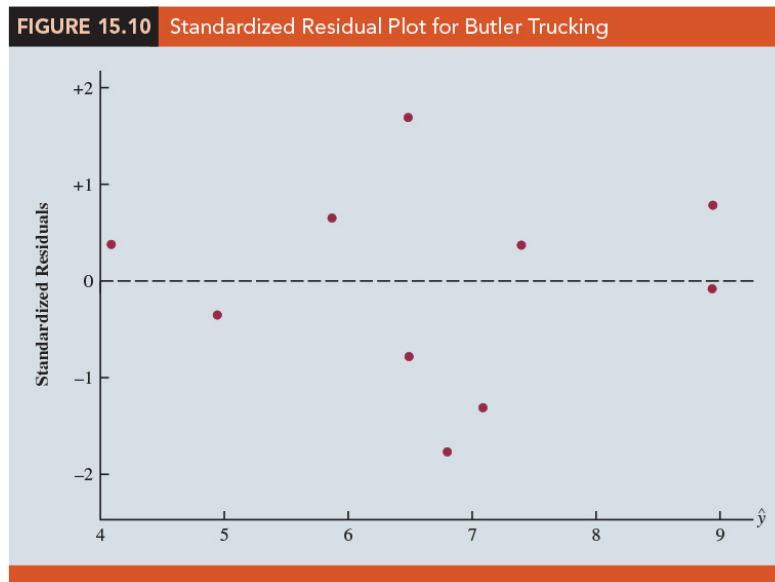
$$s_{y_i - \hat{y}_i} = \frac{s\sqrt{1 - h_i}}{\quad} \quad (15.24)$$

where s is the standard error of the estimate and h_i is the leverage of observation i . ($h_i = \mathbf{H}_{ii}$, $\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$)

3. (Chapter 14) the leverage of an observation is determined by how far the values of the independent variables are from their means.
4. The computation of h_i , $s_{y_i - \hat{y}_i}$, and hence the standardized residual for observation i in multiple regression analysis is too complex to be done by hand. However, the standardized residuals can be easily obtained as part of the output from statistical software.
5. Example Butler Trucking example
 - (a) (Table 15.7) the estimated regression equation $\hat{y} = -0.869 + 0.03821x_1 + 0.923x_2$.

Kilometers Traveled (x_1)	Deliveries (x_2)	Travel Time (y)	Predicted Value (\hat{y})	Residual ($y - \hat{y}$)	Standardized Residual
160	4	9.3	8.93846	.361541	.78344
80	3	4.8	4.95830	-.158304	-.34962
160	4	8.9	8.93846	-.038460	-.08334
160	2	6.5	7.09161	-.591609	-1.30929
80	2	4.2	4.03488	.165121	.38167
128	2	6.2	5.86892	.331083	.65431
120	3	7.4	6.48667	.913331	1.68917
104	4	6.0	6.79875	-.798749	-1.77372
144	3	7.6	7.40369	.196311	.36703
144	2	6.1	6.48026	-.380263	-.77639

- (b) (Figure 15.10) This standardized residual plot does not indicate any unusual abnormalities. All the standardized residuals are between -2 and +2; hence, we have no reason to question the assumption that the error term ϵ is normally distributed. We conclude that the model assumptions are reasonable.



- (c) (Recall Section 14.8) A normal probability plot also can be used to determine whether the distribution of ϵ appears to be normal. The same procedure is appropriate for multiple regression.

Detecting Outliers

1. An outlier is an observation that is unusual in comparison with the other data. An outlier does not fit the pattern of the other data.
2. (Chapter 14) An observation is classified as an outlier if the value of its standardized residual is less than -2 or greater than $+2$.
3. (Table 15.7) Applying this rule to the standardized residuals for the Butler Trucking example, We do not detect any outliers in the data set.
4. In general, the presence of one or more outliers in a data set tends to increase s , the standard error of the estimate, and hence increase $s_{y_i - \hat{y}_i}$, the standard deviation of residual i .
5. Because $s_{y_i - \hat{y}_i}$ appears in the denominator of the formula for the standardized residual (15.23), the size of the standardized residual will decrease as s increases. As a result, even though a residual may be unusually large, the large denominator in expression (15.23) may cause the standardized residual rule to fail to identify the observation as being an outlier.
6. We can circumvent this difficulty by using a form of the standardized residuals called studentized deleted residuals.

Studentized Deleted Residuals and Outliers

1. Suppose the i th observation is deleted from the data set and a new estimated regression equation is developed with the remaining $n-1$ observations.
2. Let $s_{(i)}$ denote the standard error of the estimate based on the data set with the i th observation deleted. If we compute the standard deviation of residual i using $s_{(i)}$ instead of s , and then compute the standardized residual for observation i using the revised $s_{y_i - \hat{y}_i}$ value, the resulting standardized residual is called a studentized deleted residual.
3. If the i th observation is an outlier, $s_{(i)}$ will be less than s . The absolute value of the i th studentized deleted residual therefore will be larger than the absolute value of the standardized residual.

4. Studentized deleted residuals may detect outliers that standardized residuals do not detect.
5. The t distribution can be used to determine whether the studentized deleted residuals indicate the presence of outliers.
 - (a) If we delete the i th observation, the number of observations in the reduced data set is $n-1$; in this case the error sum of squares has $(n-1)-p-1$ degrees of freedom.
 - (b) **Example** For the Butler Trucking example with $n = 10$ and $p = 2$, the degrees of freedom for the error sum of squares with the i th observation deleted is $9-2-1 = 6$. At $\alpha = 0.05$ level of significance, the t distribution shows that with six degrees of freedom, $t_{0.025,6} = 2.447$.
 - (c) If the value of the i th studentized deleted residual is less than -2.447 or greater than $+2.447$, we can conclude that the i th observation is an outlier.
 - (d) (Table 15.8) Butler Trucking example, outliers are not present in the data set.

TABLE 15.8 Studentized Deleted Residuals for Butler Trucking

Kilometers Traveled (x_1)	Deliveries (x_2)	Travel Time (y)	Standardized Residual	Studentized Deleted Residual
160	4	9.3	.78344	.75939
80	3	4.8	-.34962	-.32654
160	4	8.9	-.08334	-.07720
160	2	6.5	-1.30929	-1.39494
80	2	4.2	.38167	.35709
128	2	6.2	.65431	.62519
120	3	7.4	1.68917	2.03187
104	4	6.0	-1.77372	-2.21314
144	3	7.6	.36703	.34312
144	2	6.1	-.77639	-.75190

Influential Observations

1. (Section 14.9) we discussed how the leverage of an observation can be used to identify observations for which the value of the independent variable may have a strong

influence on the regression results.

2. The leverage of an observation, denoted h_i , measures how far the values of the independent variables are from their mean values.
3. We use the rule of thumb $h_i > 3(p + 1)/n$ to identify influential observations.
4. **Example** Butler Trucking example ($n = 10, p = 2$)
 - (a) The critical value for leverage is $3(2 + 1)/10 = 0.9$.
 - (b) (Table 15.9) Because h_i does not exceed 0.9, we do not detect influential observations in the data set.

TABLE 15.9 Leverage and Cook's Distance Measures for Butler Trucking

Kilometers Traveled (x_1)	Deliveries (x_2)	Travel Time (y)	Leverage (h_i)	Cook's D (D_i)
160	4	9.3	.351704	.110994
80	3	4.8	.375863	.024536
160	4	8.9	.351704	.001256
160	2	6.5	.378451	.347923
80	2	4.2	.430220	.036663
128	2	6.2	.220557	.040381
120	3	7.4	.110009	.117562
104	4	6.0	.382657	.650029
144	3	7.6	.129098	.006656
144	2	6.1	.269737	.074217

Using Cook's Distance Measure to Identify

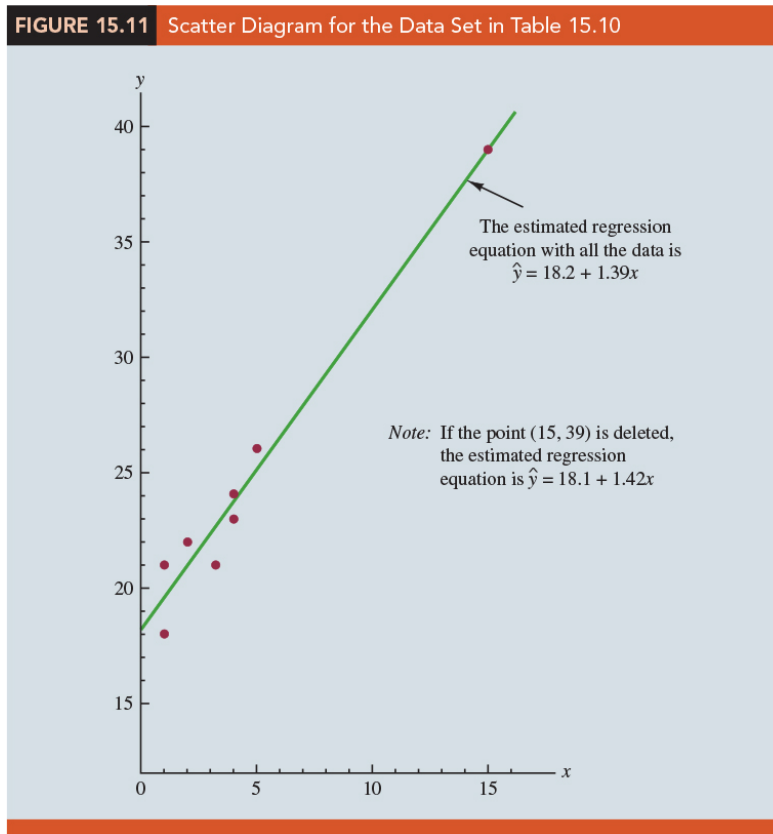
1. A problem that can arise in using leverage to identify influential observations is that an observation can be identified as having high leverage and not necessarily be influential in terms of the resulting estimated regression equation.
 - (a) (Table 15.10) Because the leverage for the eighth observation is $0.91 > 0.75$ (the critical leverage value), this observation is identified as influential.

TABLE 15.10

Data Set Illustrating
Potential Problem Using
the Leverage Criterion

x_i	y_i	Leverage h_i
1	18	.204170
1	21	.204170
2	22	.164205
3	21	.138141
4	23	.125977
4	24	.125977
5	26	.127715
15	39	.909644

(b) (Figure 15.11) the estimated regression equation: $\hat{y} = 18.2 + 1.39x$



- (c) Delete the observation $x = 15, y = 39$ from the data set and fit a new estimated regression equation to the remaining seven observations; the new estimated regression equation is $\hat{y} = 18.1 + 1.42x$
- (d) We note that the y -intercept and slope of the new estimated regression equation

are very close to the values obtained using all the data.

- (e) Although the leverage criterion identified the eighth observation as influential, this observation clearly had little influence on the results obtained. Thus, in some situations using only leverage to identify influential observations can lead to wrong conclusions.

2. **Cook' s distance measure** uses both the leverage of observation i , h_i , and the residual for observation i , $(y_i - \hat{y}_i)$, to determine whether the observation is influential.

$$D_i = \frac{(y_i - \hat{y}_i)^2}{(p + 1)s^2} \left[\frac{h_i}{(1 - h_i)^2} \right]$$

- (a) The value of Cook' s distance measure will be large and indicate an influential observation if the residual or the leverage is large.
- (b) As a rule of thumb, values of $D_i > 1$ indicate that the i th observation is influential and should be studied further.
- (c) **Example** (Table 15.9) Cook' s distance measure for the Butler Trucking problem. Observation 8 with $D_i = 0.650029 < 1$, we should not be concerned about the presence of influential observations in the Butler Trucking data set.

15.9 Logistic Regression

- In many regression applications, the dependent variable may only assume two discrete values.
- Example** A bank might want to develop an estimated regression equation for predicting whether a person will be approved for a credit card. The dependent variable can be coded as $y = 1$ if the bank approves the request for a credit card and $y = 0$ if the bank rejects the request for a credit card.
- Using logistic regression we can estimate the probability that the bank will approve the request for a credit card given a particular set of values for the chosen independent variables.

4. **Example** Simmons Stores. Let us consider an application of logistic regression involving a direct mail promotion being used by Simmons Stores.
- Simmons owns and operates a national chain of women's apparel stores. Five thousand copies of an expensive four-color sales catalog have been printed, and each catalog includes a coupon that provides a \$50 discount on purchases of \$200 or more. The catalogs are expensive and Simmons would like to send them to only those customers who have a high probability of using the coupon.
 - Management believes that annual spending at Simmons Stores and whether a customer has a Simmons credit card are two variables that might be helpful in predicting whether a customer who receives the catalog will use the coupon.
 - Simmons conducted a pilot study using a random sample of 50 Simmons credit card customers and 50 other customers who do not have a Simmons credit card. Simmons sent the catalog to each of the 100 customers selected. At the end of a test period, Simmons noted whether each customer had used her or his coupon.
 - (Table 15.11) The amount each customer spent last year at Simmons is shown in thousands of dollars and the credit card information has been coded as 1 if the customer has a Simmons credit card and 0 if not. In the Coupon column, a 1 is recorded if the sampled customer used the coupon and 0 if not.

Customer	Annual Spending (\$1000)	Simmons Card	Coupon
1	2.291	1	0
2	3.215	1	0
3	2.135	1	0
4	3.924	0	0
5	2.528	1	0
6	2.473	0	1
7	2.384	0	0
8	7.076	0	0
9	1.182	1	1
10	3.345	0	0

- We might think of building a multiple regression model using the data in Table 15.11 to help Simmons estimate whether a catalog recipient will use

the coupon. We would use Annual Spending (\$1000) and Simmons Card as independent variables and Coupon as the dependent variable.

5. Because the dependent variable may only assume the values of 0 or 1, however, the ordinary multiple regression model is not applicable. This example shows the type of situation for which logistic regression was developed.

Logistic Regression Equation

1. In multiple regression analysis, the mean or expected value of y is referred to as the multiple regression equation.

$$E(y) = \beta_0 + \beta_1x_1 + \beta_2x_2 + \cdots + \beta_px_p \quad (15.26)$$

2. (**Logistic Regression Equation**) In logistic regression, statistical theory as well as practice has shown that the relationship between $E(y)$ and x_1, x_2, \dots, x_p is better described by the following nonlinear equation.

$$E(y) = \frac{e^{\beta_0 + \beta_1x_1 + \beta_2x_2 + \cdots + \beta_px_p}}{1 + e^{\beta_0 + \beta_1x_1 + \beta_2x_2 + \cdots + \beta_px_p}} \quad (15.27)$$

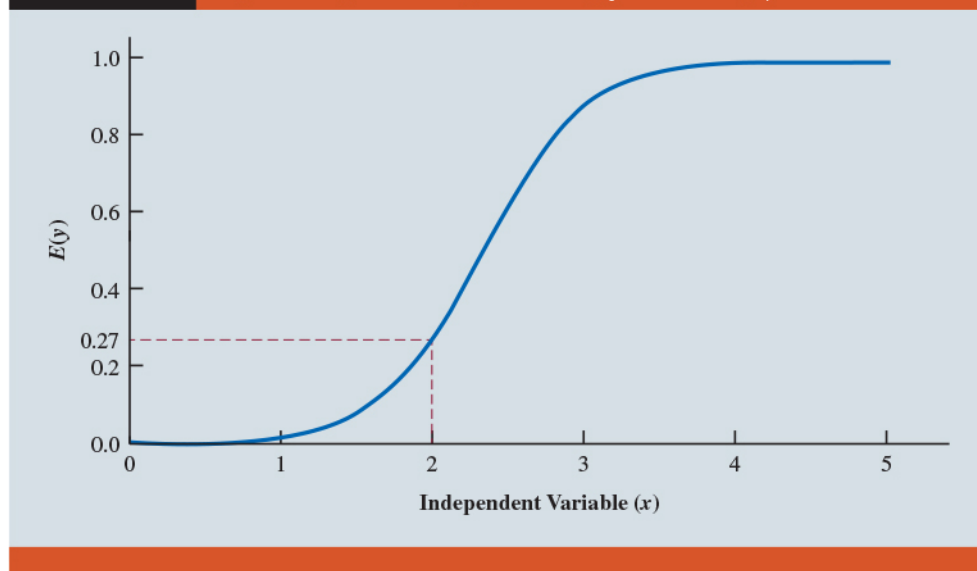
3. If the two values of the dependent variable y are coded as 0 or 1, the value of $E(y)$ in equation (15.27) provides the probability that $y = 1$ given a particular set of values for the independent variables x_1, x_2, \dots, x_p .
4. Because of the interpretation of $E(y)$ as a probability, the logistic regression equation is often written:

$$E(y) = \frac{P(y = 1|x_1, x_2, \dots, x_p)}{1 + e^{\beta_0 + \beta_1x_1 + \beta_2x_2 + \cdots + \beta_px_p}} \quad (15.28)$$

5. **Example** Suppose the model involves only one independent variable x and the values of the model parameters are $\beta_0 = -7$ and $\beta_1 = 3$. The logistic regression equation corresponding to these parameter values is

$$E(y) = \frac{P(y = 1|x) = \frac{e^{\beta_0 + \beta_1x}}{1 + e^{\beta_0 + \beta_1x}} = \frac{e^{-7+3x}}{1 + e^{-7+3x}}}{1 + e^{\beta_0 + \beta_1x}} \quad (15.29)$$

- (a) (Figure 15.12) shows a graph of equation (15.29). Note that the graph is S-shaped. The value of $E(y)$ ranges from 0 to 1.

FIGURE 15.12 Logistic Regression Equation for $\beta_0 = -7$ and $\beta_1 = 3$


- (b) For example, when $x = 2$, $E(y)$ is approximately 0.27. Also note that the value of $E(y)$ gradually approaches 1 as the value of x becomes larger and the value of $E(y)$ approaches 0 as the value of x becomes smaller.
- (c) For example, when $x = 2$, $E(y) = 0.269$. Note also that the values of $E(y)$, representing probability, increase fairly rapidly as x increases from 2 to 3. The fact that the values of $E(y)$ range from 0 to 1 and that the curve is S-shaped makes equation (15.29) ideally suited to model the probability the dependent variable is equal to 1.

Estimating the Logistic Regression Equation

1. The nonlinear form of the logistic regression equation makes the method of computing estimates more complex and beyond the scope of this text. We use statistical software to provide the estimates.
2. The estimated logistic regression equation is

$$\hat{y} = \frac{\text{estimate of } P(y = 1|x_1, x_2, \dots, x_p)}{1 + e^{b_0 + b_1x_1 + b_2x_2 + \dots + b_px_p}} = \frac{e^{b_0 + b_1x_1 + b_2x_2 + \dots + b_px_p}}{1 + e^{b_0 + b_1x_1 + b_2x_2 + \dots + b_px_p}} \quad (15.30)$$

3. Here, \hat{y} provides an estimate of the probability that $y = 1$ given a particular set of values for the independent variables.

4. **Example** Simmons Stores

(a) The variables are defined:

$$y = \begin{cases} 0 & \text{if the customer did not use the coupon} \\ 1 & \text{if the customer used the coupon} \end{cases}$$

$$x_1 = \text{annual spending at Simmons Stores (\$1000s)}$$

$$x_2 = \begin{cases} 0 & \text{if the customer does not have a Simmons credit card} \\ 1 & \text{if the customer has a Simmons credit card} \end{cases}$$

(b) Thus, we choose a logistic regression equation with two independent variables.

$$E(y) = \frac{e^{b_0 + b_1 x_1 + b_2 x_2}}{1 + e^{b_0 + b_1 x_1 + b_2 x_2}} \quad (15.31)$$

Using the sample data (see Table 15.11), we used statistical software to compute estimates of the model parameters b_0 , b_1 , and b_2 .

(c) (Figure 15.13)

FIGURE 15.13 Logistic Regression Output for the Simmons Stores Example

Significance Tests			
Term	Degrees of Freedom	χ^2	p-Value
Whole Model	2	13.63	.0011
Spending	1	7.56	.0060
Card	1	6.41	.0013
Parameter Estimates			
Term	Estimate	Standard Error	
Intercept	-2.146	.577	
Spending	.342	.129	
Card	1.099	.44	
Odds Ratios			
Term	Odds Ratio	Lower 95%	Upper 95%
Spending	1.4073	1.0936	1.8109
Card	3.0000	1.2550	7.1730

(d) We see that $\beta_0 = -2.146$, $\beta_1 = 0.342$, and $\beta_2 = 1.099$. Thus, the estimated logistic regression equation is

$$\hat{y} = \frac{e^{-2.146 + 0.342x_1 + 1.099x_2}}{1 + e^{-2.146 + 0.342x_1 + 1.099x_2}} \quad (15.32)$$

- (e) An estimate the probability of using the coupon for customers who spend \$2000 annually and do not have a Simmons credit card is approximately 0.19 ($x_1 = 2$ and $x_2 = 0$):

$$\hat{y} = \frac{e^{-2.146+0.342(2)+1.099(0)}}{1 + e^{-2.146+0.342(2)+1.099(0)}} = \frac{e^{-1.462}}{1 + e^{-1.462}} = 0.1882$$

- (f) An estimate the probability of using the coupon for customers who spent \$2000 last year and have a Simmons credit card is approximately 0.41. ($x_1 = 2$ and $x_2 = 1$):

$$\hat{y} = \frac{e^{-2.146+0.342(2)+1.099(1)}}{1 + e^{-2.146+0.342(2)+1.099(1)}} = \frac{e^{-0.363}}{1 + e^{-0.363}} = 0.4102$$

- (g) It appears that the probability of using the coupon is much higher for customers with a Simmons credit card.
- (h) Before reaching any conclusions, however, we need to assess the statistical significance of our model.

Testing for Significance

- Testing for significance in logistic regression is similar to testing for significance in multiple regression.
- First we conduct a test for overall significance. For the Simmons Stores example, the hypotheses for the test of overall significance follow:

$$H_0 : \underline{\beta_1 = \beta_2 = 0}$$

$$H_a : \text{One or both of the parameters is not equal to zero}$$

- (a) The test for overall significance is based upon the value of a χ^2 test statistic. If the null hypothesis is true, the sampling distribution of χ^2 follows a chi-square distribution with degrees of freedom equal to the number of independent variables in the model.
- (b) (Figure 15.13) The calculations of χ^2 is beyond the scope of the book. The value of χ^2 and its corresponding p -value in the Whole Model row of the Significance Tests table is 13.63 and its p -value is 0.0011. Thus, at any level of significance $\alpha \geq 0.0011$, we would reject the null hypothesis and conclude that the overall model is significant.

(c) **NOTE:**

- i. Logistic Regression: <https://online.stat.psu.edu/stat462/node/207/>
- ii. Logistic regression (Wikipedia): https://en.wikipedia.org/wiki/Logistic_regression

3. If the χ^2 test shows an overall significance, another χ^2 test can be used to determine whether each of the individual independent variables is making a significant contribution to the overall model.

(a) For the independent variables x_i , the hypotheses are

$$\underline{H_0 : \beta_i = 0 \quad H_a : \beta_i \neq 0}$$

- (b) If the null hypothesis is true, the sampling distribution of χ^2 follows a chi-square distribution with one degree of freedom.
- (c) (Figure 15.13) The Spending and Card rows of the Significance Tests table of Figure 15.13 contain the values of χ^2 and their corresponding p -values test for the estimated coefficients. Suppose we use $\alpha = 0.05$ to test for the significance of the independent variables in the Simmons model.
- (d) For the independent variable Spending (x_1) the χ^2 value is 7.56 and the corresponding p -value is 0.0060. Thus, at the 0.05 level of significance we can reject $H_0 : \beta_1 = 0$.
- (e) In a similar fashion we can also reject $H_0 : \beta_2 = 0$ because the p -value corresponding to Card's $\chi^2 = 6.41$ is 0.0013. Hence, at the 0.05 level of significance, both independent variables are statistically significant.

Managerial Use

1. We described how to develop the estimated logistic regression equation and how to test it for significance.
2. **Example** For Simmons Stores, we already computed $P(y = 1|x_1 = 2, x_2 = 1) = 0.4102$ and $P(y = 1|x_1 = 2, x_2 = 0) = 0.1881$. These probabilities indicate that for customers with annual spending of \$2000 the presence of a Simmons credit card increases the probability of using the coupon.

3. (Table 15.12) The estimated probabilities for values of annual spending ranging from \$1000 to \$7000 for both customers who have a Simmons credit card and customers who do not have a Simmons credit card.

TABLE 15.12 Estimated Probabilities for Simmons Stores

		Annual Spending						
		\$1000	\$2000	\$3000	\$4000	\$5000	\$6000	\$7000
Credit Card	Yes	.3307	.4102	.4948	.5796	.6599	.7320	.7936
	No	.1414	.1881	.2460	.3148	.3927	.4765	.5617

4. How can Simmons use this information to better target customers for the new promotion? Suppose Simmons wants to send the promotional catalog only to customers who have a 0.40 or higher probability of using the coupon. Using the estimated probabilities in Table 15.12, Simmons promotion strategy would be:
- Customers who have a Simmons credit card: Send the catalog to every customer who spent a\$2000 or more last year.
 - Customers who do not have a Simmons credit card: Send the catalog to every customer who spent \$6000 or more last year.
5. The probability of using the coupon for customers who do not have a Simmons credit card but spend \$5000 annually is 0.3922. Thus, Simmons may want to consider revising this strategy by including those customers who do not have a credit card, as long as they spent \$5000 or more last year.

Interpreting the Logistic Regression Equation

- With logistic regression, it is difficult to interpret the relation between the independent variables and the probability that $y = 1$ directly because the logistic regression equation is nonlinear.
- The relationship can be interpreted indirectly using a concept called the odds ratio (勝算比).

3. The odds (勝算) in favor of an event occurring is defined as the probability the event will occur divided by the probability the event will not occur. In logistic regression the event of interest is always $y = 1$.

4. Given a particular set of values for the independent variables, the odds in favor of $y = 1$ can be calculated as follows:

$$\text{odds} = \frac{P(y = 1|x_1, x_2, \dots, x_p)}{P(y = 0|x_1, x_2, \dots, x_p)} = \frac{P(y = 1|x_1, x_2, \dots, x_p)}{1 - P(y = 1|x_1, x_2, \dots, x_p)} \quad (15.33)$$

5. The odds ratio is the odds that $y = 1$ given that one of the independent variables has been increased by one unit (odd_{s1}) divided by the odds that $y = 1$ given no change in the values for the independent variables (odd_{s0}).

(a) **Odds Ratio**

$$\text{Odds Ratio} = \frac{odd_{s1}}{odd_{s0}} \quad (15.34)$$

(b) For example, suppose we want to compare the odds of using the coupon for customers who spend \$2000 annually and have a Simmons credit card ($x_1 = 2$ and $x_2 = 1$) to the odds of using the coupon for customers who spend \$2000 annually and do not have a Simmons credit card ($x_1 = 2$ and $x_2 = 0$).

(c) We are interested in interpreting the effect of a one-unit increase in the independent variable x_2 . In this case

$$odd_{s1} = \frac{P(y = 1|x_1 = 2, x_2 = 1)}{1 - P(y = 1|x_1 = 2, x_2 = 1)}$$

and

$$odd_{s0} = \frac{P(y = 1|x_1 = 2, x_2 = 0)}{1 - P(y = 1|x_1 = 2, x_2 = 0)}$$

(d) Previously we showed that an estimate of the probability that $y = 1$ given $x_1 = 2$ and $x_2 = 1$ is 0.4102, and an estimate of the probability that $y = 1$ given $x_1 = 2$ and $x_2 = 0$ is 0.1881. Thus,

$$\text{estimate of } odd_{s1} = \frac{0.4102}{1 - 0.4102} = 0.6956$$

and

$$\text{estimate of } odd_{s2} = \frac{0.1881}{1 - 0.1881} = 0.2318$$

The estimated odds ratio is

$$\text{estimated odds ratio} = \frac{0.6956}{0.2318} = 3.00$$

- (e) Thus, we can conclude that the estimated odds in favor of using the coupon for customers who spent \$2000 last year and have a Simmons credit card are 3 times greater than the estimated odds in favor of using the coupon for customers who spent \$2000 last year and do not have a Simmons credit card.
6. The odds ratio measures the impact on the odds of a one-unit increase in only one of the independent variables.
7. The odds ratio for each independent variable is computed while holding all the other independent variables constant. But it does not matter what constant values are used for the other independent variables. For instance, if we computed the odds ratio for the Simmons credit card variable (x_2) using \$3000, instead of \$2000, as the value for the annual spending variable (x_1), we would still obtain the same value for the estimated odds ratio (3.00). Thus, we can conclude that the estimated odds of using the coupon for customers who have a Simmons credit card are 3 times greater than the estimated odds of using the coupon for customers who do not have a Simmons credit card.
8. (Figure 15.13) the estimated odds ratios for each of the independent variables. The estimated odds ratio for Spending (x_1) is 1.4073 and the estimated odds ratio for Card (x_2) is 3.0000.
9. Let us now consider the interpretation of the estimated odds ratio for the continuous independent variable x_1 . The value of 1.4073 in the Odds Ratio column of the output tells us that the estimated odds in favor of using the coupon for customers who spent \$3000 last year is 1.4073 times greater than the estimated odds in favor of using the coupon for customers who spent \$2000 last year.
10. A unique relationship exists between the odds ratio for a variable and its corresponding regression coefficient. For each independent variable in a logistic regression equation it can be shown that

$$\underline{\text{Odds ratio} = e^{\beta_i}}$$

- (a) To illustrate this relationship, consider the independent variable x_1 in the Simmons example. The estimated odds ratio for x_1 is

$$\text{Estimated odds ratio} = e^{b_1} = e^{0.342} = 1.407$$

Similarly, the estimated odds ratio for x_2 is

$$\text{Estimated odds ratio} = e^{b_2} = e^{1.099} = 3.000$$

- (b) 補充:

$$\begin{aligned} \hat{p} &= \frac{e^{b_0+b_1x_1}}{1+e^{b_0+b_1x_1}}, & 1-\hat{p} &= \frac{1}{1+e^{b_0+b_1x_1}} \\ \ln(\hat{p}) - \ln(1-\hat{p}) &= \ln(e^{b_0+b_1x_1}) - \ln(1+e^{b_0+b_1x_1}) - \ln(1) + \ln(1+e^{b_0+b_1x_1}) \\ \ln\left(\frac{\hat{p}}{1-\hat{p}}\right) &= b_0 + b_1x_1 \\ \frac{\hat{p}}{1-\hat{p}} &= e^{b_0+b_1x_1} \\ \frac{\hat{p}}{1-\hat{p}}\Big|_{x_1=0} &= e^{b_0}, & \frac{\hat{p}}{1-\hat{p}}\Big|_{x_1=1} &= e^{b_0+b_1} \\ &\Rightarrow \text{odds ratio}\Big|_{x=1/x=0} = e^{b_1} \end{aligned}$$

11. The odds ratio for an independent variable represents the change in the odds for a one unit change in the independent variable holding all the other independent variables constant.

- (a) Suppose that we want to consider the effect of a change of more than one unit, say c units. For instance, suppose in the Simmons example that we want to compare the odds of using the coupon for customers who spend \$5000 annually ($x_1 = 5$) to the odds of using the coupon for customers who spend \$2000 annually ($x_1 = 2$). In this case $c = 5 - 2 = 3$ and the corresponding estimated odds ratio is

$$\underline{e^{cb_1} = e^{3(0.342)} = e^{1.026} = 2.79}$$

- (b) This result indicates that the estimated odds of using the coupon for customers who spend \$5000 annually is 2.79 times greater than the estimated odds of using the coupon for customers who spend \$2000 annually.

- (c) In other words, the estimated odds ratio for an increase of \$3000 in annual spending is 2.79.
- (d) In general, the odds ratio enables us to compare the odds for two different events. If the value of the odds ratio is 1, the odds for both events are the same. Thus, if the independent variable we are considering (such as Simmons credit card status) has a positive impact on the probability of the event occurring, the corresponding odds ratio will be greater than 1.
12. (Figure 15.13) Most statistical software packages provide a confidence interval for the odds ratio. The Odds Ratio table in Figure 15.13 provides a 95% confidence interval for each of the odds ratios.
- (a) For example, the point estimate of the odds ratio for x_1 is 1.4073 and the 95% confidence interval is 1.0936 to 1.8109. Because the confidence interval does not contain the value of 1, we can conclude that x_1 has a significant relationship with the estimated odds ratio.
- (b) Similarly, the 95% confidence interval for the odds ratio for x_2 is 1.2550 to 7.1730. Because this interval does not contain the value of 1, we can also conclude that x_2 has a significant relationship with the odds ratio.

Logit Transformation

1. It can be shown that

$$\ln(\text{odds}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p$$

2. This equation shows that the natural logarithm of the odds in favor of $y = 1$ is a linear function of the independent variables. This linear function is called the logit. We will use the notation $g(x_1, x_2, \dots, x_p)$ to denote the logit.

3. Logit

$$g(x_1, x_2, \dots, x_p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p \quad (15.35)$$

4. Substituting $g(x_1, x_2, \dots, x_p)$ for $\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p$ in equation (15.27), we can write the logistic regression equation as

$$E(y) = \frac{e^{g(x_1, x_2, \dots, x_p)}}{1 + e^{g(x_1, x_2, \dots, x_p)}} \quad (15.36)$$

5. Once we estimate the parameters in the logistic regression equation, we can compute an estimate of the logit. Using $\hat{g}(x_1, x_2, \dots, x_p)$ to denote the estimated logit, we obtain

$$\text{Estimated Logit } \hat{g}(x_1, x_2, \dots, x_p) = b_0 + b_1x_1 + b_2x_2 + \dots + b_px_p \quad (15.37)$$

6. Thus, in terms of the estimated logit, the estimated regression equation is

$$\hat{y} = \frac{e^{b_0 + b_1x_1 + b_2x_2 + \dots + b_px_p}}{1 + e^{b_0 + b_1x_1 + b_2x_2 + \dots + b_px_p}} = \frac{e^{\hat{g}(x_1, x_2, \dots, x_p)}}{1 + e^{\hat{g}(x_1, x_2, \dots, x_p)}} \quad (15.38)$$

7. For the Simmons Stores example, the estimated logit is

$$\hat{g}(x_1, x_2) = -2.146 + 0.342x_1 + 1.099x_2$$

and the estimated regression equation is

$$\hat{y} = \frac{e^{\hat{g}(x_1, x_2)}}{1 + e^{\hat{g}(x_1, x_2)}} = \frac{e^{-2.146 + 0.342x_1 + 1.099x_2}}{1 + e^{-2.146 + 0.342x_1 + 1.099x_2}}$$

Thus, because of the unique relationship between the estimated logit and the estimated logistic regression equation, we can compute the estimated probabilities for Simmons Stores by dividing $e^{\hat{g}(x_1, x_2)}$ by $1 + e^{\hat{g}(x_1, x_2)}$.

☺ **EXERCISES**

15.2 : 1, 5, 6

15.3 : 11, 14, 15

15.5 : 19, 23, 24

15.6 : 27, 29

15.7 : 32, 34, 35

15.8 : 40, 41

15.9 : 44, 46, 48

SUP : 51, 55.

“你無法改變別人的長相，但我們可以改變我們看人的方式。”

“You can not change someone’s looks, but we can change the way we look.”

— 奇蹟男孩 (*Wonder*, 2017)

統計學 (二)

Anderson's Statistics for Business & Economics (14/E)

Chapter 17: Time Series Analysis and Forecasting

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17.1 Time Series Patterns

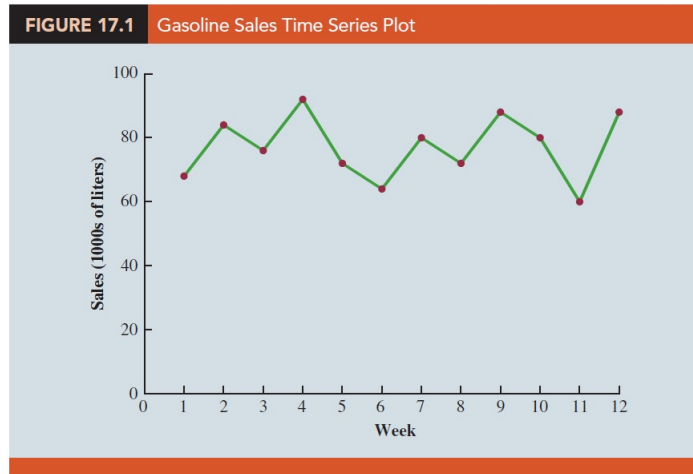
1. **time series**: A time series is a sequence of observations on a variable measured at successive points in time or over successive periods of time.
2. The measurements may be taken every hour, day, week, month, or year, or at any other regular interval. (this textbook limits the discussion to time series in which the values of the series are recorded at equal intervals)
3. The pattern of the data is an important factor in understanding how the time series has behaved in the past. If such behavior can be expected to continue in the future, we can use the past pattern to guide us in selecting an appropriate forecasting method.
4. A time series plot is a graphical presentation of the relationship between time and the time series variable; time is on the horizontal axis and the time series values are shown on the vertical axis. A time series plot is useful to identify the underlying pattern in the data.
5. Some of the common types of data patterns that can be identified when examining a time series plot: horizontal pattern, trend pattern, seasonal pattern, trend and seasonal pattern, and cyclical pattern.

Horizontal Pattern

1. A horizontal pattern exists when the data fluctuate around a constant mean.
2. **Example** (Table 17.1) (Figure 17.1) These data show the number of gallons of gasoline sold by a gasoline distributor in Bennington, Vermont, over the past 12 weeks.

TABLE 17.1

Gasoline Sales Time Series	
Week	Sales (1000s of gallons)
1	17
2	21
3	19
4	23
5	18
6	16
7	20
8	18
9	22
10	20
11	15
12	22



The average value or mean for this time series is 19.25 gallons (1000s) per week. Although random variability is present, we would say that these data follow a horizontal pattern.

3. The term stationary time series is used to denote a time series whose statistical properties are independent of time.
4. In particular this means that
 - (a) The process generating the data has a constant mean.
 - (b) The variability of the time series is constant over time.
5. A time series plot for a stationary time series will always exhibit a horizontal pattern. But simply observing a horizontal pattern is not sufficient evidence to conclude that the time series is stationary.
6. More advanced texts on forecasting discuss procedures for determining if a time series is stationary and provide methods for transforming a time series that is not stationary into a stationary series.

7. Changes in business conditions can often result in a time series that has a horizontal pattern shifting to a new level.

- (a) **Example** For instance, suppose the gasoline distributor signs a contract with the Vermont State Police to provide gasoline for state police cars located in southern Vermont. With this new contract, the distributor expects to see a major increase in weekly sales starting in week 13.
- (b) (Table 17.2) The number of gallons of gasoline sold for the original time series and for the 10 weeks after signing the new contract.

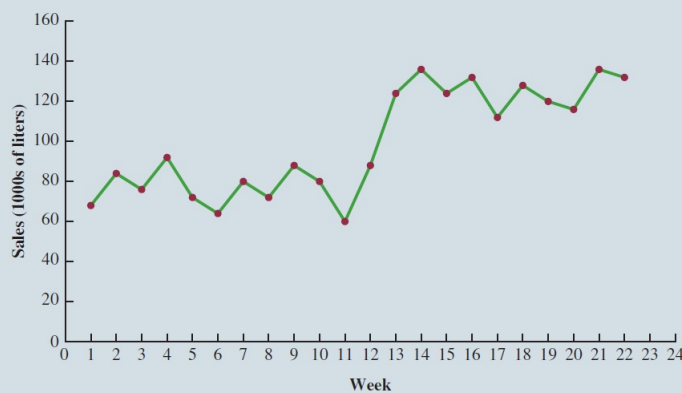
TABLE 17.2

Gasoline Sales Time Series After Obtaining the Contract with the Vermont State Police

Week	Sales (1000s of liters)
1	68
2	84
3	76
4	92
5	72
6	64
7	80
8	72
9	88
10	80
11	60
12	88
13	124
14	136
15	124
16	132
17	112
18	128
19	120
20	116
21	136
22	132

FIGURE 17.2

Gasoline Sales Time Series Plot After Obtaining the Contract with the Vermont State Police



- (c) (Figure 17.2) Note the increased level of the time series beginning in week 13. This change in the level of the time series makes it more difficult to choose an appropriate forecasting method.

8. Selecting a forecasting method that adapts well to changes in the level of a time series is an important consideration in many practical applications.

Trend Pattern

1. Although time series data generally exhibit random fluctuations, a time series may also show gradual shifts or movements to relatively higher or lower values over a longer period of time.
2. If a time series plot exhibits this type of behavior, we say that a trend pattern exists.
3. A trend is usually the result of long-term factors such as population increases or decreases, changing demographic characteristics of the population, technology, and/or consumer preferences.
4. **Example** (Table 17.3) (Figure 17.3) Consider the time series of bicycle sales for a particular manufacturer over the past 10 years.

TABLE 17.3
Bicycle Sales Time Series

Year	Sales (1000s)
1	21.6
2	22.9
3	25.5
4	21.9
5	23.9
6	27.5
7	31.5
8	29.7
9	28.6
10	31.4



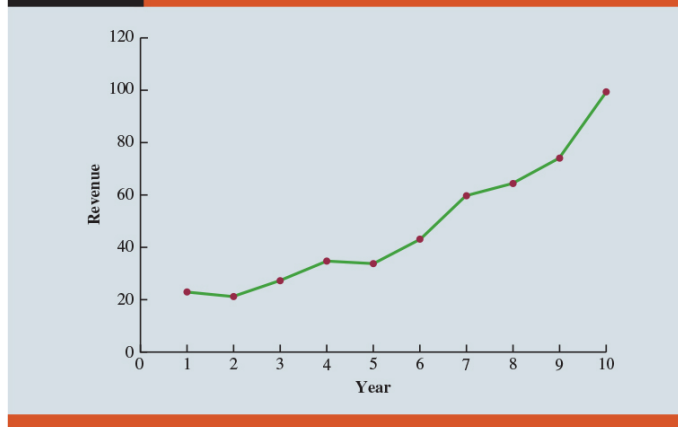
Visual inspection of the time series plot shows some up and down movement over the past 10 years, but the time series also seems to have a systematically increasing or upward trend. The trend for the bicycle sales time series appears to be linear and increasing over time.

5. **Example** (Table 17.4) (Figure 17.4) The data show the sales for a cholesterol drug since the company won FDA approval for it 10 years ago.

TABLE 17.4

Cholesterol Revenue Time Series (\$Millions)	
Year	Revenue
1	23.1
2	21.3
3	27.4
4	34.6
5	33.8
6	43.2
7	59.5
8	64.4
9	74.2
10	99.3

FIGURE 17.4 Cholesterol Revenue Times Series Plot (\$Millions)



The time series increases in a nonlinear fashion; that is, the rate of change of revenue does not increase by a constant amount from one year to the next. In fact, the revenue appears to be growing in an exponential fashion.

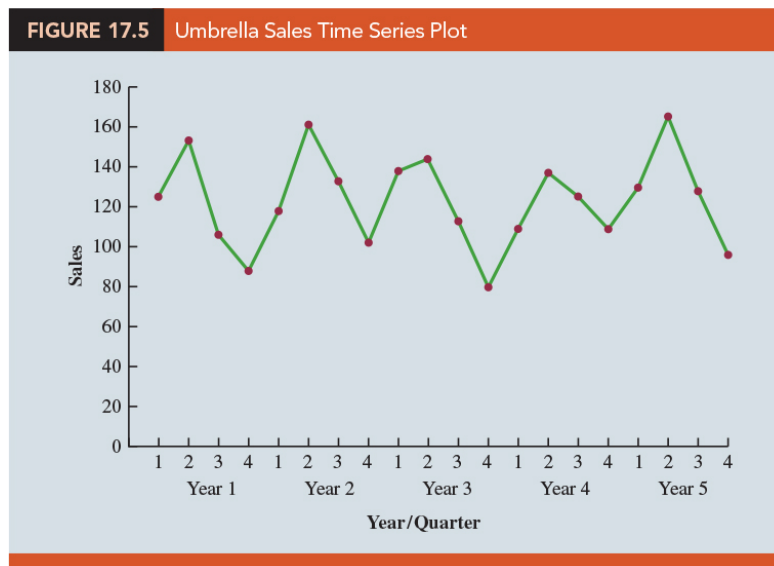
- Exponential relationships such as this are appropriate when the percentage change from one period to the next is relatively constant.

Seasonal Pattern

- The trend of a time series can be identified by analyzing multiyear movements in historical data. Seasonal patterns are recognized by seeing the same repeating patterns over successive periods of time.
- Example** For example, a manufacturer of swimming pools expects low sales activity in the fall and winter months, with peak sales in the spring and summer months. Manufacturers of snow removal equipment and heavy clothing, however, expect just the opposite yearly pattern.
- The pattern for a time series plot that exhibits a repeating pattern over a one-year period due to seasonal influences is called a seasonal pattern.
- Example** Daily traffic volume shows within-the-day "seasonal" behavior, with peak levels occurring during rush hours, moderate flow during the rest of the day and early evening, and light flow from midnight to early morning.

5. **Example** (Table 17.5) (Figure 17.5) As an example of a seasonal pattern, consider the number of umbrellas sold at a clothing store over the past five years.

Year	Quarter	Sales
1	1	125
	2	153
	3	106
	4	88
2	1	118
	2	161
	3	133
	4	102
3	1	138
	2	144
	3	113
	4	80
4	1	109
	2	137
	3	125
	4	109
5	1	130
	2	165
	3	128
	4	96



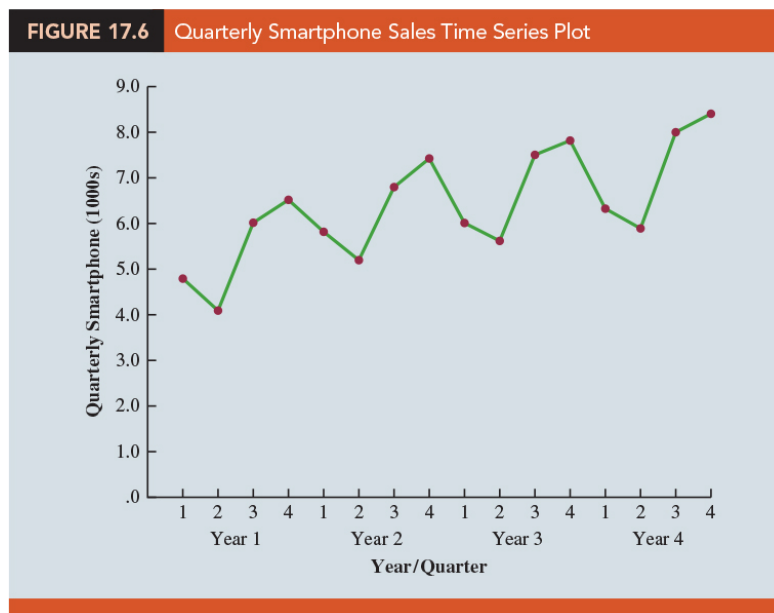
The time series plot does not indicate any long-term trend in sales. The data follow a horizontal pattern. But closer inspection of the time series plot reveals a regular pattern in the data. That is, the first and third quarters have moderate sales, the second quarter has the highest sales, and the fourth quarter tends to have the lowest sales volume. Thus, we would conclude that a quarterly seasonal pattern is present.

Trend and Seasonal Pattern

1. Some time series include a combination of a trend and seasonal pattern.
2. **Example** (Table 17.6) (Figure 17.6) The smartphone sales for a particular manufacturer over the past four years.

TABLE 17.6 Quarterly Smartphone Sales Time Series

Year	Quarter	Sales (1000s)
1	1	4.8
	2	4.1
	3	6.0
	4	6.5
2	1	5.8
	2	5.2
	3	6.8
	4	7.4
3	1	6.0
	2	5.6
	3	7.5
	4	7.8
4	1	6.3
	2	5.9
	3	8.0
	4	8.4

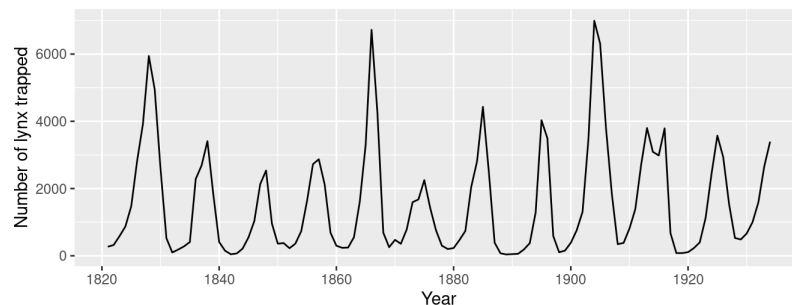


3. Clearly, an increasing trend is present.

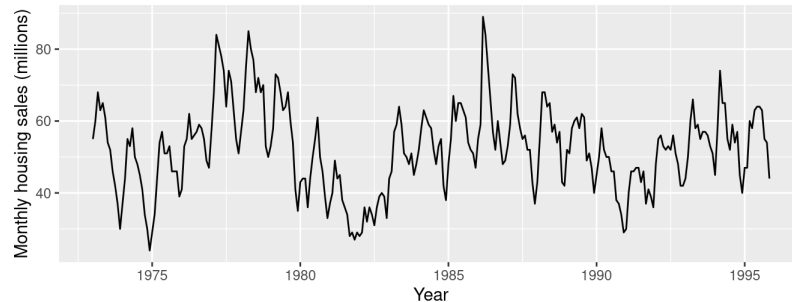
4. But, Figure 17.6 also indicates that sales are lowest in the second quarter of each year and increase in quarters 3 and 4. Thus, we conclude that a seasonal pattern also exists for smartphone sales.
5. In such cases we need to use a forecasting method that has the capability to deal with both trend and seasonality.

Cyclical Pattern

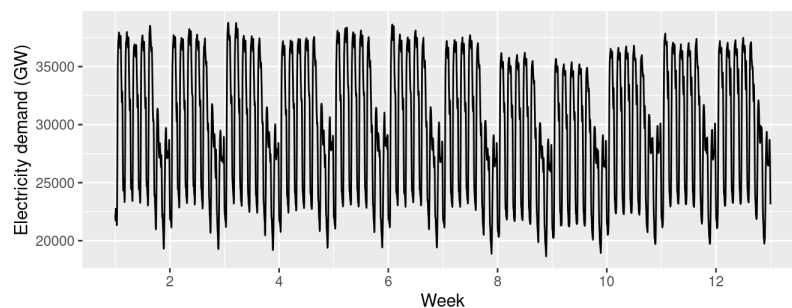
1. A cyclical pattern exists if the time series plot shows an alternating sequence of points below and above the trend line lasting more than one year.
2. Often, the cyclical component of a time series is due to multiyear business cycles.
3. **Example** For example, periods of moderate inflation followed by periods of rapid inflation can lead to time series that alternate below and above a generally increasing trend line (e.g., a time series for housing costs).
4. A cyclical pattern repeats with some regularity over several years. Cyclical patterns differ from seasonal patterns in that cyclical patterns occur over multiple years, whereas seasonal patterns occur within one year.
5. **More Example** <https://robjhyndman.com/hyndsight/cyclicts/>
 - (a) The plot shows the famous Canadian lynx (山貓) data –the number of lynx trapped each year in the McKenzie (麥肯錫) river district of northwest Canada (1821-1934). These show clear aperiodic (非週期性的) population cycles of approximately 10 years. The cycles are not of fixed length –some last 8 or 9 years and others last longer than 10 years.



- (b) The plot shows the monthly sales of new one-family houses sold in the USA (1973-1995). There is strong seasonality within each year, as well as some strong cyclic behaviour with period about 6–10 years.



- (c) The plot shows half-hourly electricity demand in England and Wales from Monday 5 June 2000 to Sunday 27 August 2000. Here there are two types of seasonality – a daily pattern and a weekly pattern. If we collected data over a few years, we would also see there is an annual pattern. If we collected data over a few decades, we may even see a longer cyclic pattern.



6. Business cycles are extremely difficult, if not impossible, to forecast. As a result, cyclical effects are often combined with long-term trend effects and referred to as trend-cycle effects.

Selecting a Forecasting Method

1. The underlying pattern in the time series is an important factor in selecting a forecasting method. Thus, a time series plot should be one of the first things developed when trying to determine which forecasting method to use.

2. The next two sections illustrate methods that can be used in situations where the underlying pattern is horizontal; in other words, no trend or seasonal effects are present. We then consider methods appropriate when trend and/or seasonality are present in the data.

17.2 Forecast Accuracy

1. The simplest of all the forecasting methods (a naive forecasting method): an approach that uses the most recent week's sales volume as the forecast for the next week.
2. (Table 17.7) The distributor sold 68 thousand gallons of gasoline in week 1; this value is used as the forecast for week 2. Next, we use 84, the actual value of sales in week 2, as the forecast for week 3, and so on.

TABLE 17.7 Computing Forecasts and Measures of Forecast Accuracy Using the Most Recent Value as the Forecast for the Next Period

Week	Time Series Value	Forecast	Forecast Error	Absolute Value of Forecast Error	Squared Forecast Error	Percentage Error	Absolute Value of Percentage Error
1	68						
2	84	68	16	16	256	19.05	19.05
3	76	84	-8	8	64	-10.53	10.53
4	92	76	16	16	256	17.39	17.39
5	72	92	-20	20	400	-27.78	27.78
6	64	72	-8	8	64	-12.50	12.50
7	80	64	16	16	256	20.00	20.00
8	72	80	-8	8	64	-11.11	11.11
9	88	72	16	16	256	18.18	18.18
10	80	88	-8	8	64	-10.00	10.00
11	60	80	-20	20	400	-33.33	33.33
12	88	60	28	28	784	31.82	31.82
		Totals	20	164	2864	1.19	211.69

3. The key concept associated with measuring forecast accuracy is forecast error, defined as

$$\text{Forecast Error} = \text{Actual Value} - \text{Forecast}$$

- (a) **Example** Because the distributor actually sold 84 thousand gallons of gasoline in week 2 and the forecast, using the sales volume in week 1, was 68 thousand gallons, the forecast error in week 2 is

$$\text{Forecast Error in week 2} = \underline{84 - 68 = 16}$$

- (b) The fact that the forecast error is positive indicates that in week 2 the forecasting method underestimated the actual value of sales. Next, we use 84, the actual value of sales in week 2, as the forecast for week 3. Since the actual value of sales in week 3 is 76, the forecast error for week 3 is $76 - 84 = -8$. In this case, the negative forecast error indicates that in week 3 the forecast overestimated the actual value.
- (c) A simple measure of forecast accuracy is the mean or average of the forecast errors. Table 17.7 shows that the sum of the forecast errors for the gasoline sales time series is 20; thus, the mean or average forecast error is $20/11 = 1.82$.
- (d) Because positive and negative forecast errors tend to offset one another, the mean error is likely to be small; thus, the mean error is not a very useful measure of forecast accuracy.
4. The mean absolute error, denoted MAE, is the average of the absolute values of the forecast errors.
- (a) MAE is a measure of forecast accuracy that avoids the problem of positive and negative forecast errors offsetting one another.
- (b) (Table 17.7) the sum of the absolute values of the forecast errors is 164:

$$\text{MAE} = \text{average of the absolute value of forecast errors} = \underline{164/11=14.91}$$

5. Another measure that avoids the problem of positive and negative forecast errors offsetting each other is obtained by computing the average of the squared forecast errors (mean squared error, is denoted MSE.)

$$\text{MSE} = \text{average of the sum of squared forecast errors} = \underline{2864/11=260.36}$$

6. The size of MAE and MSE depends upon the scale of the data. As a result, it is difficult to make comparisons for different time intervals, such as comparing

a method of forecasting monthly gasoline sales to a method of forecasting weekly sales, or to make comparisons across different time series.

7. The mean absolute percentage error, denoted MAPE, is a percentage error corresponding to the forecast of 84 in week 2 is computed by dividing the forecast error in week 2 by the actual value in week 2 and multiplying the result by 100.

(a) For week 2 the percentage error is computed as follows:

$$\text{Percentage error for week 2} = \frac{\text{forecast error}}{\text{actual value}} = 16/84 \times (100) = 19.05\%$$

Thus, the forecast error for week 2 is 19.05% of the observed value in week 2.

(b) The sum of the absolute values of the percentage errors is 211.69:

$$\begin{aligned} \text{MAPE} &= \text{average of the absolute value of percentage forecast errors} \\ &= \frac{211.69}{11} = 19.24\% \end{aligned}$$

8. Summarizing, using the naive (most recent observation) forecasting method, we obtained the following measures of forecast accuracy:

$$\text{MAE} = 3.73, \quad \text{MSE} = 16.27, \quad \text{MAPE} = 19.24\%$$

9. These measures of forecast accuracy simply measure how well the forecasting method is able to forecast historical values of the time series.
10. Suppose we want to forecast sales for a future time period, such as week 13. In this case the forecast for week 13 is 88, the actual value of the time series in week 12. Is this an accurate estimate of sales for week 13? Unfortunately, there is no way to address the issue of accuracy associated with forecasts for future time periods. But, if we select a forecasting method that works well for the historical data, and we think that the historical pattern will continue into the future, we should obtain results that will ultimately be shown to be good.
11. (Table 17.8) Suppose we use the average of all the historical data available as the forecast for the next period. We begin by developing a forecast for week 2. Since there is only one historical value available prior to week 2, the forecast for week 2

is just the time series value in week 1; thus, the forecast for week 2 is 84 thousand gallons of gasoline. To compute the forecast for week 3, we take the average of the sales values in weeks 1 and 2. Thus,

$$\text{Forecast for week 3} = \frac{(68 + 84)}{2} = 76$$

TABLE 17.8 Computing Forecasts and Measures of Forecast Accuracy Using the Average of All the Historical Data as the Forecast for the Next Period

Week	Time Series Value	Forecast	Forecast Error	Absolute Value of Forecast Error	Squared Forecast Error	Percentage Error	Absolute Value of Percentage Error
1	68						
2	84	68.00	16.00	16.00	256.00	19.05	19.05
3	76	76.00	.00	.00	.00	.00	.00
4	92	76.00	16.00	16.00	256.00	17.39	17.39
5	72	80.00	-8.00	8.00	64.00	-11.11	11.11
6	64	78.40	-14.40	14.40	207.36	-22.50	22.50
7	80	76.00	4.00	4.00	16.00	5.00	5.00
8	72	76.57	-4.57	4.57	20.90	-6.35	6.35
9	88	76.00	12.00	12.00	144.00	13.64	13.64
10	80	77.33	2.67	2.67	7.11	3.33	3.33
11	60	77.60	-17.60	17.60	309.76	-29.33	29.33
12	88	76.00	12.00	12.00	144.00	13.64	13.64
		Totals	18.10	107.24	1425.13	2.76	141.34

12. Comparing the values of MAE, MSE, and MAPE for each method:

	Naive Method	Average of Past Values
MAE	14.91	9.75
MSE	260.36	129.56
MAPE	19.24%	12.85%

13. For every measure, the average of past values provides more accurate forecasts than using the most recent observation as the forecast for the next period.

14. In general, if the underlying time series is stationary, the average of all the historical data will always provide the best results.

(a) (Recall Table 17.2) But suppose that the underlying time series is not stationary. Note the change in level in week 13 for the resulting time series. When a shift to a new level like this occurs, it takes a long time for the forecasting method that uses the average of all the historical data to adjust to the new level of the time series.

- (b) In this case, the simple naive method adjusts very rapidly to the change in level because it uses the most recent observation available as the forecast.
- (c) Measures of forecast accuracy are important factors in comparing different forecasting methods, but we have to be careful not to rely upon them too heavily.
- (d) Good judgment and knowledge about business conditions that might affect the forecast also have to be carefully considered when selecting a method. And historical forecast accuracy is not the only consideration, especially if the time series is likely to change in the future.

17.3 Moving Averages and Exponential Smoothing

1. Three forecasting methods that are appropriate for a time series with a horizontal pattern: moving averages, weighted moving averages, and exponential smoothing.
2. The objective of each of these methods is to smooth out the random fluctuations in the time series, they are referred to as smoothing methods.
3. These methods are easy to use and generally provide a high level of accuracy for short-range forecasts, such as a forecast for the next time period.

Moving Averages

1. (**Moving Average Forecast of Order k**) The moving averages method uses the average of the most recent k data values in the time series as the forecast for the next period:

$$F_{t+1} = \frac{\sum (\text{most recent } k \text{ data values})}{k} = \frac{Y_t + Y_{t-1} + \cdots + Y_{t-(k-1)}}{k} \quad (17.1)$$

where F_{t+1} is the forecast of the times series for period $t + 1$ and Y_t is the actual value of the time series in period t .

2. The average will change, or move, as new observations become available.
- To use moving averages to forecast a time series, we must first select the order, or number of time series values, to be included in the moving average.
 - If only the most recent values of the time series are considered relevant, a small value of k is preferred.
 - If more past values are considered relevant, then a larger value of k is better.
 - A time series with a horizontal pattern can shift to a new level over time. A moving average will adapt to the new level of the series and resume providing good forecasts in k periods.
 - Thus, a smaller value of k will track shifts in a time series more quickly. But larger values of k will be more effective in smoothing out the random fluctuations over time.

3. **Example** (Recall Table 17.1 and Figure 17.1) the gasoline sales data

- The time series plot in Figure 17.1 indicates that the gasoline sales time series has a horizontal pattern. Thus, the smoothing methods of this section are applicable.
- Use a three-week moving average ($k = 3$), the forecast of sales in week 4 using the average of the time series values in weeks 1–3:

$$F_4 = \text{average of weeks 1-3} = \frac{69 + 84 + 76}{3} = 76$$

Thus, the moving average forecast of sales in week 4 is 76 or 76,000 liters of gasoline.

- The actual value observed in week 4 is 92, the forecast error in week 4 is $92 - 76 = 16$.
- (Table 17.9) The forecast of sales in week 5 by averaging the time series values in weeks 2–4.

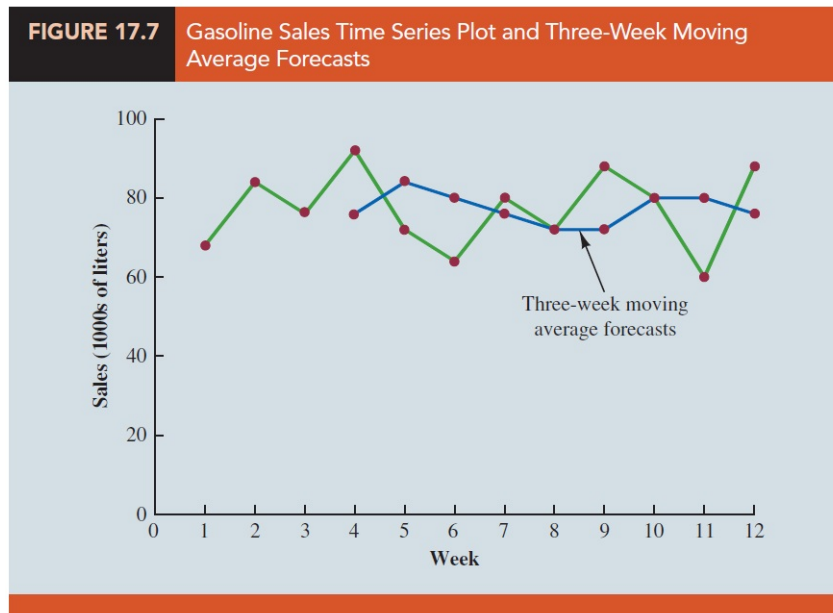
$$F_5 = \text{average of weeks 2-4} = \frac{84 + 76 + 92}{3} = 84$$

Hence, the forecast of sales in week 5 is 84 and the error associated with this forecast is $72 - 84 = -12$.

TABLE 17.9 Summary of Three-Week Moving Average Calculations

Week	Time Series Value	Forecast	Forecast Error	Absolute Value of Forecast Error	Squared Forecast Error	Percentage Error	Absolute Value of Percentage Error
1	68						
2	84						
3	76						
4	92	76	16	16	256	17.39	17.39
5	72	84	-12	12	144	-16.67	16.67
6	64	80	-16	16	256	-25.00	25.00
7	80	76	4	4	16	5.00	5.00
8	72	72	0	0	0	.00	.00
9	88	72	16	16	256	18.18	18.18
10	80	80	0	0	0	.00	.00
11	60	80	-20	20	400	-33.33	33.33
12	88	76	12	12	144	13.64	13.64
		Totals	0	96	1472	-20.79	129.21

- (e) (Figure 17.7) Note how the graph of the moving average forecasts has tended to smooth out the random fluctuations in the time series.



- (f) To forecast sales in week 13, the next time period in the future, we simply compute the average of the time series values in weeks 10, 11, and 12.

$$F_{13} = \text{average of weeks 10-12} = \frac{80 + 60 + 88}{3} = 76$$

- (g) **Forecast Accuracy** Using the three-week moving average calculations in Table 17.9, the values for these three measures of forecast accuracy (MAE, MSE,

and MAPE) are

$$\begin{aligned}\text{MAE} &= \frac{96}{9} = 10.67 \quad (\text{mean absolute error}) \\ \text{MSE} &= \frac{1472}{9} = 163.56 \quad (\text{mean squared error}) \\ \text{MAPE} &= \frac{129.21}{9} = 14.36\% \quad (\text{mean absolute percentage error})\end{aligned}$$

- (h) (Recall Section 17.2) Using the most recent observation as the forecast for the next week (a moving average of order $k = 1$) resulted in values of $\text{MAE} = 14.91$, $\text{MSE} = 260.36$, and $\text{MAPE} = 19.24\%$. Thus, in each case the three-week moving average approach provided more accurate forecasts than simply using the most recent observation as the forecast.
- To determine if a moving average with a different order k can provide more accurate forecasts, we recommend using trial and error to determine the value of k that minimizes MSE.
 - For the gasoline sales time series, it can be shown that the minimum value of MSE corresponds to a moving average of order $k = 6$ with $\text{MSE} = 6.79$. If we are willing to assume that the order of the moving average that is best for the historical data will also be best for future values of the time series, the most accurate moving average forecasts of gasoline sales can be obtained using a moving average of order $k = 6$.

Weighted Moving Averages

- In the moving averages method, each observation in the moving average calculation receives the same weight.
- One variation, known as weighted moving averages, involves selecting a different weight for each data value and then computing a weighted average of the most recent k values as the forecast.
- In most cases, the most recent observation receives the most weight, and the weight decreases for older data values.

4. A moving average forecast of order $k = 3$ is just a special case of the weighted moving averages method in which each weight is equal to $1/3$. Note that for the weighted moving average method the sum of the weights is equal to 1.
5. **Example** We assign a weight of $3/6$ to the most recent observation, a weight of $2/6$ to the second most recent observation, and a weight of $1/6$ to the third most recent observation. Using this weighted average, our forecast for week 4 is:

$$\text{Forecast for week 4} = \frac{1}{6}(68) + \frac{2}{6}(84) + \frac{3}{6}(76) = 77.33$$

6. To use the weighted moving averages method, we must first select the number of data values to be included in the weighted moving average and then choose weights for each of the data values. In general, if we believe that the recent past is a better predictor of the future than the distant past, larger weights should be given to the more recent observations. However, when the time series is highly variable, selecting approximately equal weights for the data values may be best.
7. **Forecast Accuracy** To determine whether one particular combination of number of data values and weights provides a more accurate forecast than another combination, we recommend using MSE as the measure of forecast accuracy. That is, if we assume that the combination that is best for the past will also be best for the future, we would use the combination of number of data values and weights that minimizes MSE for the historical time series to forecast the next value in the time series.

Exponential Smoothing

1. Exponential smoothing also uses a weighted average of past time series values as a forecast; it is a special case of the weighted moving averages method in which we select only one weight —the weight for the most recent observation.
2. The weights for the other data values are computed automatically and become smaller as the observations move farther into the past.

3. Exponential Smoothing Forecast

$$F_{t+1} = \frac{\alpha Y_t + (1 - \alpha)F_t}{\quad} \quad (17.2)$$

where

F_{t+1} : forecast of the time series for period $(t + 1)$

Y_t : actual value of the time series in period t

F_t : forecast of the time series for period t

α : smoothing constant ($0 \leq \alpha \leq 1$)

4. Equation (17.2) shows that the forecast for period $t + 1$ is a weighted average of the actual value in period t and the forecast for period t .
5. The weight given to the actual value in period t is the smoothing constant α and the weight given to the forecast in period t is $1 - \alpha$.
6. Let us illustrate by working with a time series involving only three periods of data: Y_1 , Y_2 , and Y_3 .

- (a) To initiate the calculations, we let F_1 equal the actual value of the time series in period 1; that is, $F_1 = Y_1$. Hence, the forecast for period 2 is

$$F_2 = \frac{\alpha Y_1 + (1 - \alpha)F_1}{\quad} = \frac{\alpha Y_1 + (1 - \alpha)Y_1}{\quad} = Y_1$$

We see that the exponential smoothing forecast for period 2 is equal to the actual value of the time series in period 1.


- (b) The forecast for period 3 is

$$F_3 = \frac{\alpha Y_2 + (1 - \alpha)F_2}{\quad} = \frac{\alpha Y_2 + (1 - \alpha)Y_1}{\quad}$$

- (c) Finally, substituting this expression for F_3 in the expression for F_4 , we obtain

$$\begin{aligned} F_4 &= \alpha Y_3 + (1 - \alpha)F_3 \\ &= \alpha Y_3 + (1 - \alpha)[\alpha Y_2 + (1 - \alpha)Y_1] \\ &= \frac{\alpha Y_3 + \alpha(1 - \alpha)Y_2 + (1 - \alpha)^2 Y_1}{\quad} \end{aligned}$$

- (d) We now see that F_4 is a weighted average of the first three time series values. The sum of the coefficients, or weights, for Y_1 , Y_2 , and Y_3 equals 1.
- (e) A similar argument can be made to show that, in general, any forecast F_{t+1} is a weighted average of all the previous time series values.

 Question (p876)

Use exponential smoothing approach with a smoothing parameter $\alpha = 0.2$ to obtain F_2, F_3, F_4 and F_{13} for the gasoline sales time series in Table 17.1 and Figure 17.1. Start the calculations, set the exponential smoothing forecast for period 2 equal to the actual value of the time series in period 1.

sol:

TABLE 17.10 Summary of the Exponential Smoothing Forecasts and Forecast Errors for the Gasoline Sales Time Series with Smoothing Constant $\alpha = .2$

Week	Time Series Value	Forecast	Forecast Error	Squared Forecast Error
1	68			
2	84	68.00	16.00	256.00
3	76	71.20	4.80	23.04
4	92	72.16	19.84	393.63
5	72	76.13	-4.13	17.06
6	64	75.30	-11.30	127.69
7	80	73.04	6.96	48.44
8	72	74.43	-2.43	5.90
9	88	73.95	14.05	197.40
10	80	76.76	3.24	10.50
11	60	77.41	-17.41	303.11
12	88	73.92	14.08	198.25
		Totals	43.70	1581.02

FIGURE 17.8 Actual and Forecast Gasoline Sales Time Series with Smoothing Constant $\alpha = .2$

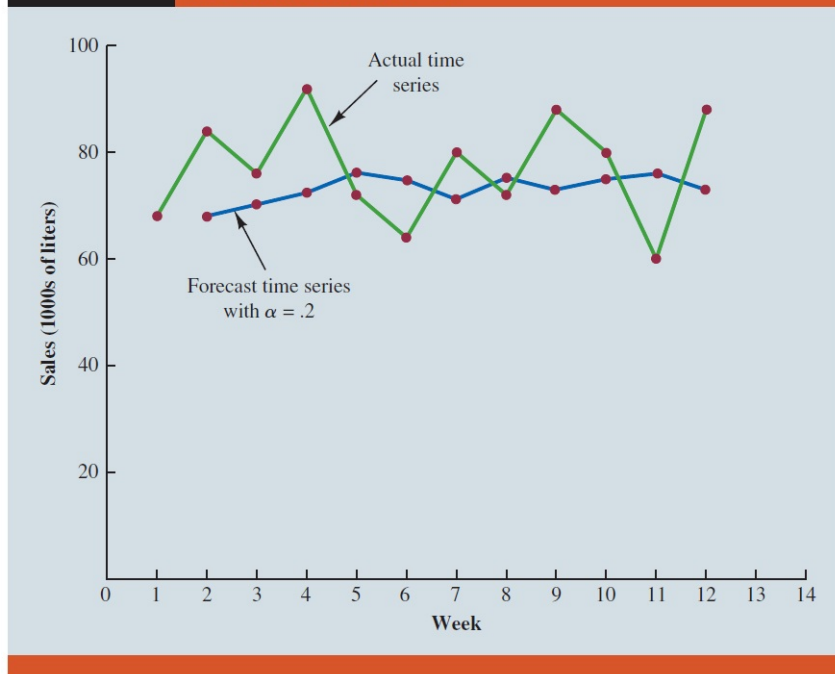


TABLE 17.11 Summary of the Exponential Smoothing Forecasts and Forecast Errors for the Gasoline Sales Time Series with Smoothing Constant $\alpha = .3$

Week	Time Series Value	Forecast	Forecast Error	Squared Forecast Error
1	68			
2	84	68.00	16.00	256.00
3	76	72.80	3.20	10.24
4	92	73.76	18.24	332.70
5	72	79.23	-7.23	52.27
6	64	77.06	-13.06	170.56
7	80	73.14	6.86	47.06
8	72	75.20	-3.20	10.24
9	88	74.24	13.76	189.34
10	80	78.37	1.63	2.66
11	60	78.86	-18.86	355.70
12	88	73.20	14.80	219.04
		Totals	32.14	1645.81

1. **Forecast Accuracy** (Table 17.10)(Figure 17.8)(Table 17.11) The criterion we will use to determine a desirable value for the smoothing constant α is the same as the criterion we proposed for determining the order or number of periods of data to include in the moving averages calculation. That is, we choose the value of α that minimizes the MSE.
2. The exponential smoothing results with $\alpha = 0.2$: the value of the sum of squared forecast errors is 98.80; hence $MSE = 98.80/11 = 8.98$. The exponential smoothing results with $\alpha = 0.3$: the value of the sum of squared forecast errors is 102.83; hence $MSE = 102.83/11 = 9.35$.
3. Thus, we would be inclined to prefer the original smoothing constant of $\alpha = 0.2$. Using a trial-and-error calculation with other values of α , we can find a "good" value for the smoothing constant.

17.4 Trend Projection

1. We present two forecasting methods in this section that are appropriate for time series exhibiting a trend pattern.

- (a) First, we show how simple linear regression can be used to forecast a time series with a linear trend.
- (b) Next we show how the curve-fitting capability of regression analysis can also be used to forecast time series with a curvilinear or nonlinear trend.

Linear Trend Regression

1. (Table 17.12) (Figure 17.9) the bicycle sales time series: the linear trend line provides a reasonable approximation of the long-run movement in the series.

TABLE 17.12
Bicycle Sales Time Series

Year	Sales (1000s)
1	21.6
2	22.9
3	25.5
4	21.9
5	23.9
6	27.5
7	31.5
8	29.7
9	28.6
10	31.4



2. The estimated regression equation describing a straight-line relationship between an independent variable x and a dependent variable y is written as

$$\hat{y} = b_0 + b_1x$$

where \hat{y} is the estimated or predicted value of y .

3. To emphasize the fact that in forecasting the independent variable is time, we will replace x with t and \hat{y} with T_t to emphasize that we are estimating the trend for a time series.

4. Linear Trend Equation

$$T_t = b_0 + b_1t \quad (17.4)$$

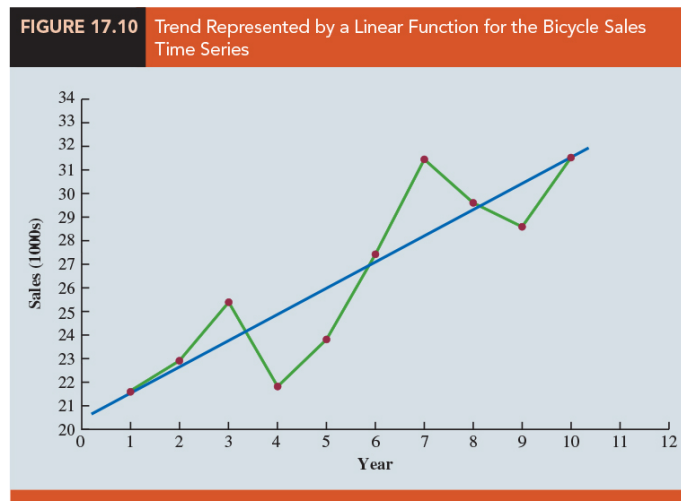
where

T_t = linear trend forecast in period t

b_0 = intercept of the linear trend line

b_1 = slope of the linear trend line

t = time period, $t = 1$ ($t = n$) corresponding to the first time (most recent) series observation



5. Computing the Slope and Intercept for a Linear Trend

$$b_1 = \frac{\sum_{t=1}^n (t - \bar{t})(Y_t - \bar{Y})}{\sum (t - \bar{t})^2}$$

$$= \frac{\sum tY_t - n\bar{t}\bar{Y}}{\sum t^2 - n\bar{t}^2} \quad (17.5)$$

$$b_0 = \frac{\bar{Y} - b_1\bar{t}}{1} \quad (17.6)$$

where

Y_t = value of the time series in period t

n = number of time periods (number of observations)

\bar{Y} = average value of the time series

\bar{t} = average value of t

6. Example the bicycle sales time series

TABLE 17.13 Summary of Linear Trend Calculations for the Bicycle Sales Time Series

t	Y_t	$t - \bar{t}$	$Y_t - \bar{Y}$	$(t - \bar{t})(Y_t - \bar{Y})$	$(t - \bar{t})^2$
1	21.6	-4.5	-4.85	21.825	20.25
2	22.9	-3.5	-3.55	12.425	12.25
3	25.5	-2.5	-.95	2.375	6.25
4	21.9	-1.5	-4.55	6.825	2.25
5	23.9	-.5	-2.55	1.275	.25
6	27.5	.5	1.05	.525	.25
7	31.5	1.5	5.05	7.575	2.25
8	29.7	2.5	3.25	8.125	6.25
9	28.6	3.5	2.15	7.525	12.25
10	31.4	4.5	4.95	22.275	20.25
Totals	55	264.5		90.750	82.50

(a) The linear trend equation is

$$T_t = \underline{20.4 + 1.1t}$$

(b) The slope of 1.1 indicates that over the past 10 years the firm experienced an average growth in sales of about 1100 units per year.

(c) If we assume that the past 10-year trend in sales is a good indicator of the future, this trend equation can be used to develop forecasts for future time periods. For example, the next year's trend projection or forecast, T_{11} .

$$T_{11} = \underline{20.4 + 1.1(11) = 32.5}$$

Thus, using trend projection, we would forecast sales of 32,500 bicycles next year.

7. (Table 17.14) Use MSE to compute the accuracy associated with the trend projection forecasting method.

TABLE 17.14 Summary of the Linear Trend Forecasts and Forecast Errors for the Bicycle Sales Time Series

Year	Sales (1000s) Y_t	Forecast T_t	Forecast Error	Squared Forecast Error
1	21.6	21.5	.1	.01
2	22.9	22.6	.3	.09
3	25.5	23.7	1.8	3.24
4	21.9	24.8	-2.9	8.41
5	23.9	25.9	-2.0	4.00
6	27.5	27.0	.5	.25
7	31.5	28.1	3.4	11.56
8	29.7	29.2	.5	.25
9	28.6	30.3	-1.7	2.89
10	31.4	31.4	.0	.00
			Total	30.70

$$MSE = \frac{\sum_{t=1}^n (Y_t - F_t)^2}{n} = \frac{30.7}{10} = 3.07$$

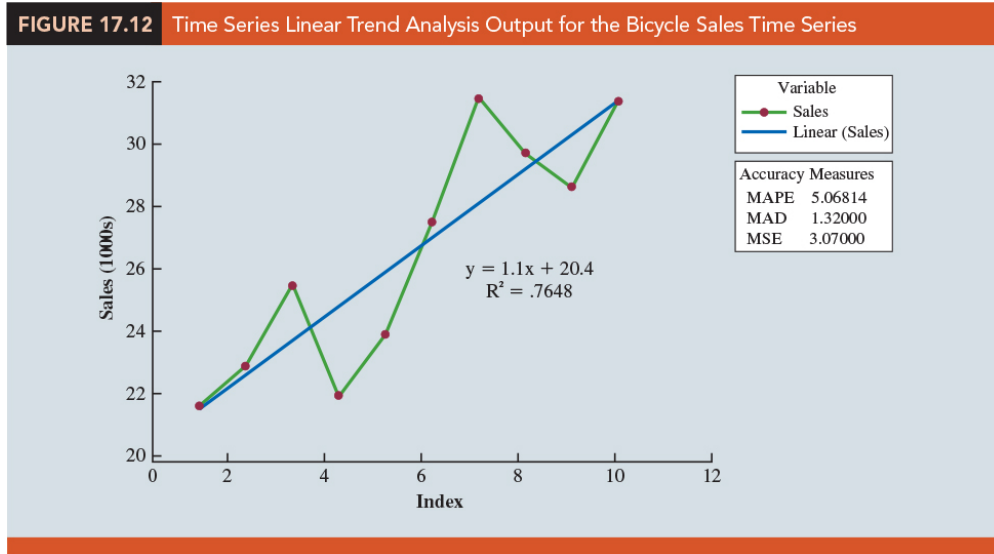
8. Because linear trend regression in forecasting is the same as the standard regression analysis procedure applied to time-series data, we can use statistical software to perform the calculations.

9. (Figure 17.11) the value of MSE in the ANOVA table is

$$MSE = \frac{\text{Sum of Squares Due to Error}}{\text{Degrees of Freedom}} = \frac{30.7}{8} = 3.837$$

FIGURE 17.11 Regression Output for the Bicycle Sales Time Series

Regression Analysis: Sales Versus Year					
Analysis of Variance					
Source	DF	Adj SS	Adj MS	F-Value	P-Value
Regression	1	99.82	99.825	26.01	.001
Error	8	30.70	3.837		
Total	9	130.52			
Model Summary					
	S	R-sq			
	1.95895	76.48%			
Coefficients					
Term	Coef	SE Coef	T-Value	P-Value	
Constant	20.40	1.34	15.24	.000	
Year	1.10	.216	5.10	.001	
Regression Equation					
Sales = 20.40 + 1.100 Year					



- This value of MSE differs from the value of MSE that we computed previously because the sum of squared errors is divided by 8 instead of 10; thus, MSE in the regression output is not the average of the squared forecast errors.
- NOTE:** Most forecasting packages, however, compute MSE by taking the average of the squared errors. Thus, when using time series packages to develop a trend equation, the value of MSE that is reported may differ slightly from the value you would obtain using a general regression approach.

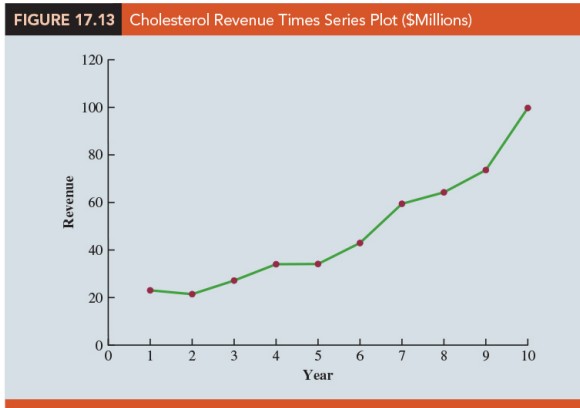
Nonlinear Trend Regression

- Example** (Table 17.15) (Figure 17.13) Consider the annual revenue in millions of dollars for a cholesterol drug for the first 10 years of sales.

TABLE 17.15

Cholesterol Revenue Time Series (\$ Millions)

Year (t)	Revenue (\$ millions)
1	23.1
2	21.3
3	27.4
4	34.6
5	33.8
6	43.2
7	59.5
8	64.4
9	74.2
10	99.3



2. The time series plot indicates an overall increasing or upward trend. A curvilinear function appears to be needed to model the long-term trend.
3. **Quadratic Trend Equation** A variety of nonlinear functions can be used to develop an estimate of the trend for the cholesterol time series. For instance, consider the following quadratic trend equation:

$$\underline{T_t = b_0 + b_1t + b_2t^2} \quad (17.7)$$

4. (Figure 17.14) a portion of the multiple regression output for the quadratic trend model;

FIGURE 17.14 Quadratic Trend Regression Output for the Cholesterol Revenue Time Series

Regression Analysis: Revenue Versus Year, YearSq					
Analysis of Variance					
Source	DF	Adj SS	Adj MS	F-Value	P-Value
Regression	2	5770.13	2885.06	182.52	.000
Error	7	110.65	15.81		
Total	9	5880.78			
Model Summary					
	S	R-sq			
	3.97578	98.12%			
Coefficients					
Term	Coef	SE Coef	T-Value	P-Value	
Constant	24.18	4.68	5.17	.001	
Year	-2.11	1.95	-1.08	.317	
YearSq	.922	.173	5.33	.001	
Regression Equation					
Revenue = 24.18 - 2.11 Year + .922 YearSq					

The estimated regression equation is

$$\text{Revenue (\$millions)} = 24.18 - 2.11\text{Year} + 0.922\text{YearSq}$$

5. Exponential Trend Equation

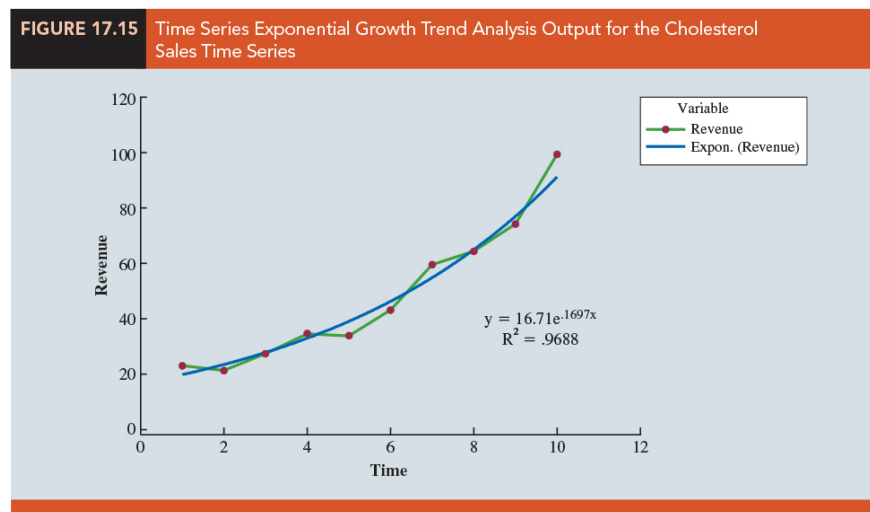
$$\underline{T_t = b_0e^{b_1t}} \quad (17.8)$$

6. Suppose $b_0 = 16.71$, and $b_1 = 0.1697$, T_t is not increasing by a constant amount as in the case of the linear trend model but by a constant percentage.

7. In this exponential trend model, multiplicative factor is $e^{0.1697(1)} = 1.185$, so the constant percentage increase from time period to time period is 18.5% .
8. Many statistical software packages have the capability to compute an exponential trend equation directly. Some software packages only provide linear trend, but by applying a natural *log* transformation to both sides of the equality in equation (17.8) we can apply the equivalent linear form:

$$\ln T_t = \ln b_0 + b_1 t$$

(Figure 17.15)



17.5 Seasonality and Trend

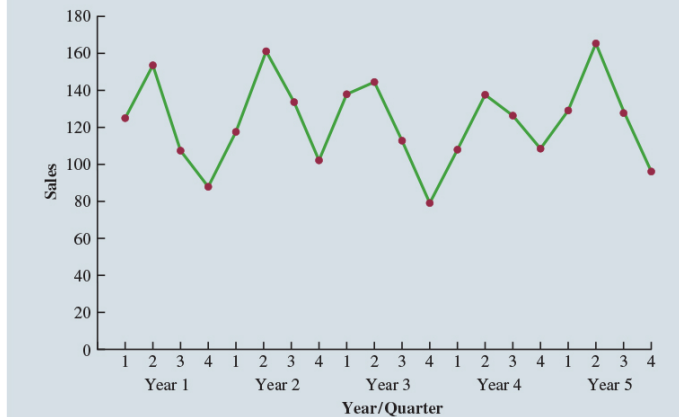
Seasonality Without Trend

1. **Example** (Table 17.16)(Figure 17.16) Consider the number of umbrellas sold at a clothing store over the past five years.

TABLE 17.16

Umbrella Sales Time Series		
Year	Quarter	Sales
1	1	125
	2	153
	3	106
	4	88
2	1	118
	2	161
	3	133
	4	102
3	1	138
	2	144
	3	113
	4	80
4	1	109
	2	137
	3	125
	4	109
5	1	130
	2	165
	3	128
	4	96

FIGURE 17.16 Umbrella Sales Time Series Plot



2. The time series plot does not indicate any long-term trend in sales. The first and third quarters have moderate sales, the second quarter has the highest sales, and the fourth quarter tends to be the lowest quarter in terms of sales volume. Thus, we would conclude that a quarterly seasonal pattern is present.
3. Just like using dummy variables to deal with an independent variable in a standard regression analysis, we can use the same approach to model a time series with a seasonal pattern by treating the season as a categorical variable.
4. Recall that when a categorical variable has k levels, $k-1$ dummy variables are required. Thus, to model the seasonal effects in the umbrella time series we need $4-1 = 3$ dummy variables:

$$Qtr1 = \begin{cases} 1 & \text{if Quarter 1} \\ 0 & \text{otherwise} \end{cases}, Qtr2 = \begin{cases} 1 & \text{if Quarter 2} \\ 0 & \text{otherwise} \end{cases}, Qtr3 = \begin{cases} 1 & \text{if Quarter 3} \\ 0 & \text{otherwise} \end{cases}$$

5. Using \hat{Y} to denote the estimated or forecasted value of sales, the general form of the estimated regression equation relating the number of umbrellas sold to the quarter

the sales take place:

$$\hat{Y} = b_0 + b_1Qtr1 + b_2Qtr2 + b_3Qtr3$$

6. (Table 17.17) the umbrella sales time series with the coded values of the dummy variables.

Year	Quarter	Qtr1	Qtr2	Qtr3	Sales
1	1	1	0	0	125
	2	0	1	0	153
	3	0	0	1	106
	4	0	0	0	88
2	1	1	0	0	118
	2	0	1	0	161
	3	0	0	1	133
	4	0	0	0	102
3	1	1	0	0	138
	2	0	1	0	144
	3	0	0	1	113
	4	0	0	0	80
4	1	1	0	0	109
	2	0	1	0	137
	3	0	0	1	125
	4	0	0	0	109
5	1	1	0	0	130
	2	0	1	0	165
	3	0	0	1	128
	4	0	0	0	96

7. (Figure 17.17) the computer output: the estimated multiple regression equation obtained is

$$Sales = 95.00 + 29.00 Qtr1 + 57.00 Qtr2 + 26.00 Qtr3$$

We can use this equation to forecast quarterly sales for next year.

Quarter 1: $Sales = 95.0 + 29.0(1) + 57.0(0) + 26.0(0) = 124.$

Quarter 2: $Sales = 95.0 + 29.0(0) + 57.0(1) + 26.0(0) = 152.$

Quarter 3: $Sales = 95.0 + 29.0(0) + 57.0(0) + 26.0(1) = 121.$

Quarter 4: $Sales = 95.0 + 29.0(0) + 57.0(1) + 26.0(0) = 95.$

Term	Coef	SE Coef	T-Value	P-Value
Constant	95.00	5.06	18.76	.000
Qtr1	29.00	7.16	4.05	.001
Qtr2	57.00	7.16	7.96	.000
Qtr3	26.00	7.16	3.63	.002

Regression Equation

$$Sales = 95.00 + 29.00 Qtr1 + 57.00 Qtr2 + 26.00 Qtr3$$

8. The regression output shown in Figure 17.17 provides additional information that can be used to assess the accuracy of the forecast and determine the significance of the results.

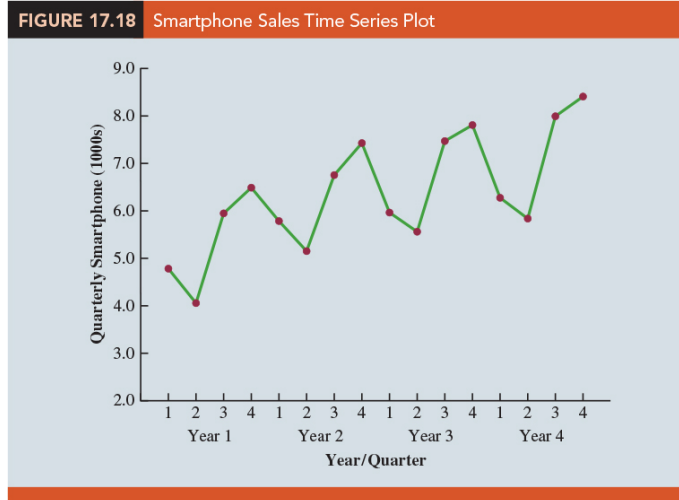
Seasonality and Trend

1. **Example** (Table 17.18) (Figure 17.18) The quarterly smartphone sales.

TABLE 17.18

Smartphone Sales Time Series

Year	Quarter	Sales (1000s)
1	1	4.8
	2	4.1
	3	6.0
	4	6.5
2	1	5.8
	2	5.2
	3	6.8
	4	7.4
3	1	6.0
	2	5.6
	3	7.5
	4	7.8
4	1	6.3
	2	5.9
	3	8.0
	4	8.4



2. The sales are lowest in the second quarter of each year and increase in quarters 3 and 4. Thus, we conclude that a seasonal pattern exists for smartphone sales.
3. But the time series also has an upward linear trend that will need to be accounted for in order to develop accurate forecasts of quarterly sales.
4. This is easily handled by combining the dummy variable approach for seasonality with the time series regression approach for handling linear trend.
5. The general form of the estimated multiple regression equation for modeling both the quarterly seasonal effects and the linear trend in the smartphone time series:

$$\hat{Y}_t = b_0 + b_1 Qtr1 + b_2 Qtr2 + b_3 Qtr3 + b_4 t$$

where

\hat{Y}_t = estimate or forecast of sales in time period t

$Qtr1 = 1$ ($Qtr2 = 1$)($Qtr3 = 1$) if time period t corresponds to the first (second) (third) quarter of the year; 0 otherwise.

6. (Table 17.19) revised smartphone sales time series that includes the coded values of the dummy variables and the time period t .

Year	Quarter	Qtr1	Qtr2	Qtr3	Period	Sales (1000s)
1	1	1	0	0	1	4.8
	2	0	1	0	2	4.1
	3	0	0	1	3	6.0
	4	0	0	0	4	6.5
2	1	1	0	0	5	5.8
	2	0	1	0	6	5.2
	3	0	0	1	7	6.8
	4	0	0	0	8	7.4
3	1	1	0	0	9	6.0
	2	0	1	0	10	5.6
	3	0	0	1	11	7.5
	4	0	0	0	12	7.8
4	1	1	0	0	13	6.3
	2	0	1	0	14	5.9
	3	0	0	1	15	8.0
	4	0	0	0	16	8.4

7. (Figure 17.19) The estimated multiple regression equation is

$$Sales(1000s) = 6.069 - 1.363 Qtr1 - 2.034 Qtr2 - 0.304Qtr3 + 0.1456 t \quad (17.9)$$

8. Forecast for Time Period 17 (Quarter 1 in Year 5):

$$Sales(1000s) = 6.069 + 1.363(1) + 2.034(0) + 0.304(0) + 0.1456(17) = 7.18$$

Thus, accounting for the seasonal effects and the linear trend in smartphone sales, the estimates of quarterly sales in year 5 are 7180, 6660, 8530, and 8980.

9. The dummy variables in the estimated multiple regression equation actually provide four estimated multiple regression equations, one for each quarter. If time period t corresponds to quarter 1, the estimate of quarterly sales is

$$\begin{aligned} Quarter1 : Sales &= 6.069 - 1.363(1) - 2.034(0) - 0.304(0) + 0.1456(t) \\ &= 4.71 + 0.1456t \end{aligned}$$

$$Quarter2 : Sales = 4.04 + 0.1456t$$

$$Quarter3 : Sales = 5.77 + 0.1456t$$

$$Quarter4 : Sales = 6.07 + 0.1456t$$

10. The slope of the trend line for each quarterly forecast equation is 0.1456, indicating a growth in sales of about 146 sets per quarter.
11. The intercept for the Quarter 1 equation is 4.71 and the intercept for Quarter 4 equation is 6.07. Thus, sales in Quarter 1 are $4.71 - 6.07 = -1.36$ or 1360 sets less than in Quarter 4.
12. The estimated regression coefficient for *Qtr1* in equation (17.9) provides an estimate of the difference in sales between Quarter 1 and Quarter 4.
13. Similar interpretations can be provided for -2.03, the estimated regression coefficient for dummy variable *Qtr2*, and -0.304, the estimated regression coefficient for dummy variable *Qtr3*.

Models Based on Monthly Data

1. For monthly data, season is a categorical variable with 12 levels and thus $12-1 = 11$ dummy variables are required.

$$Month1 = \begin{cases} 1 & \text{if January} \\ 0 & \text{otherwise} \end{cases}, \dots, Month11 = \begin{cases} 1 & \text{if November} \\ 0 & \text{otherwise} \end{cases}$$

2. Other than this change, the multiple regression approach for handling seasonality remains the same.

17.6 Time Series Decomposition*

😊 SUPPLEMENTARY EXERCISES: 41, 44, 47

☺ **EXERCISES**

17.2 : 1, 4

17.3 : 5, 9, 11, 14

17.4 : 17, 20, 22, 26

17.5 : 28, 30, 33

SUP : 41, 44, 47

“有時候壞事是注定要發生，而我們卻無能為力。那我們何必擔心呢？”

“Look, sometimes bad things happen —and there’ s nothing you can do about it. So why worry?”

— 獅子王 (*The Lion King*, 2019)

統計學 (二)

Anderson's Statistics for Business & Economics (14/E)

Chapter 18: Nonparametric Methods

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Overview

1. The statistical methods for inference presented previously are parametric methods.
2. The parametric methods begin with an assumption about the probability distribution of the population which is often that the population has a normal distribution.
3. Based upon this assumption, statisticians are able to derive the sampling distribution that can be used to make inferences about one or more parameters of the population, such as the population mean or the population standard deviation.
 - (a) (Recall Chapter 9) An inference about a population mean that was based on an assumption that the population had a normal probability distribution with unknown parameters μ and σ .
 - (b) Using the sample standard deviation s to estimate the population standard deviation σ .
 - (c) The test statistic for making an inference about the population mean was shown to have a t distribution.

- (d) The t distribution was used to compute confidence intervals and conduct hypothesis tests about the mean of a normally distributed population.
4. In this chapter we present nonparametric methods which can be used to make inferences about a population without requiring an assumption about the specific form of the population's probability distribution.
- (a) (First section) how the binomial distribution uses two categories of data to make an inference about a population median.
- (b) (Next three sections) how rank-ordered data are used in nonparametric tests about two or more populations.
- (c) (Final section) use rank-ordered data to compute the rank correlation for two variables.
5. For this reason, these nonparametric methods are also called distribution-free methods.
6. The computations used in the nonparametric methods are generally done with categorical data. Whenever the data are quantitative, we will transform the data into categorical data in order to conduct the nonparametric test.

18.1 Sign Test

Hypothesis Test About a Population Median

1. The sign test provides a nonparametric procedure for testing a hypothesis about the value of a population median.
2. If we consider a population where no data value is exactly equal to the median, the median is the measure of central tendency that divides the population so that 50% of the values are greater than the median and 50% of the values are less than the median.

3. Whenever a population distribution is skewed, the median is often preferred over the mean as the best measure of central location for the population.
4. **Example** The weekly sales of Cape May Potato Chips by the Lawler Grocery Store chain.
- (a) Lawler's management made the decision to carry the new potato chip product based on the manufacture's estimate that the median sales should be \$450 per week on a per store basis.
- (b) (Table 18.1) After carrying the product for three-months, Lawler's management requested the following hypothesis test about the population median weekly sales:

$$H_0 : \text{Median} = 450$$

$$H_a : \text{Median} \neq 450$$

Store Number	Weekly Sales (\$)	Store Number	Weekly Sales (\$)
56	485	63	474
19	562	39	662
36	415	84	380
128	860	102	515
12	426	44	721

- (c) (Table 18.2) In conducting the sign test, we compare each sample observation to the hypothesized value of the population median.
- If the observation is greater than the hypothesized value, we record a plus sign "+".
 - If the observation is less than the hypothesized value, we record a minus sign "-".
 - If an observation is exactly equal to the hypothesized value, the observation is eliminated from the sample and the analysis proceeds with the smaller sample size, using only the observations where a plus sign or a minus sign has been recorded.

Store Number	Weekly Sales (\$)	Sign	Store Number	Weekly Sales (\$)	Sign
56	485	+	63	474	+
19	562	+	39	662	+
36	415	-	84	380	-
128	860	+	102	515	+
12	426	-	44	721	+

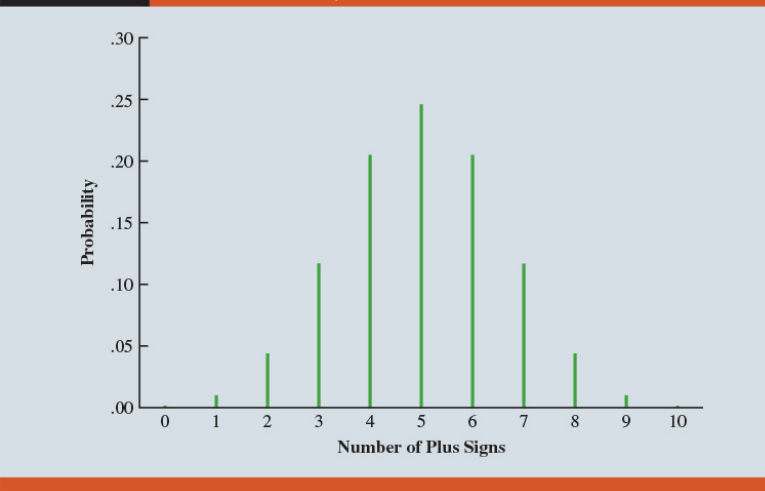
- (d) Note that there are 7 plus signs and 3 minus signs.
5. The assigning of the plus signs and minus signs has made the situation a binomial distribution application. The sample size $n = 10$ is the number of trials. There are two outcomes possible per trial, a plus sign or a minus sign, and the trials are independent. Let p denote the probability of a plus sign.
6. If the population median is 450, p would equal 0.50 as there should be 50% plus signs and 50% minus signs in the population. Thus, in terms of the binomial probability p , the sign test hypotheses about the population median are converted to the following hypotheses about the binomial probability p .

$$\begin{array}{l} H_0 : \text{Median} = 450 \\ H_a : \text{Median} \neq 450 \end{array} \Rightarrow \begin{array}{l} H_0 : p = 0.5 \\ H_a : p \neq 0.5 \end{array}$$

- (a) If H_0 cannot be rejected, we cannot conclude that p is different from 0.50 and thus we cannot conclude that the population median is different from 450.
- (b) If H_0 is rejected, we can conclude that p is not equal to 0.50 and thus the population median is not equal to 450.
7. (Table 5 in Appendix B)(Table 18.3)(Figure 18.1) With $n = 10$ stores or trials and $p = 0.50$, obtain the binomial probabilities for the number of plus signs under the assumption H_0 is true. ($X \sim B(n = 10, p = 0.5)$)

TABLE 18.3 Binomial Probabilities with $n = 10$ and $p = .50$

Number of Plus Signs	Probability
0	.0010
1	.0098
2	.0439
3	.1172
4	.2051
5	.2461
6	.2051
7	.1172
8	.0439
9	.0098
10	.0010

FIGURE 18.1 Binomial Sampling Distribution for the Number of Plus Signs When $n = 10$ and $p = .50$ 

- (a) Use a 0.10 level of significance for the test.
- (b) Since the observed number of plus signs for the sample data, 7, is in the upper tail of the binomial distribution, we compute the probability of obtaining 7 or more plus signs

$$P(X \geq 7) = 0.1172 + 0.0439 + 0.0098 + 0.0010 = 0.1719$$

- (c) Since we are using a two-tailed hypothesis test, this upper tail probability is doubled to obtain the $p\text{-value} = 2(0.1719) = 0.3438$.
- (d) With $p\text{-value} > \alpha$, we cannot reject H_0 . In terms of the binomial probability p , we cannot reject $H_0 : p = 0.50$, and thus we cannot reject the hypothesis that the population median is \$450.

8. The one-tailed sign tests about a population median:

(a) Formulated the hypotheses as an upper tail test :

$$H_0 : \text{Median} \leq 450$$

$$H_a : \text{Median} > 450$$

(b) The corresponding p -value is equal to the binomial probability that the number of plus signs is greater than or equal to 7 found in the sample.

(c) This one-tailed p -value: $0.1172 + 0.0439 + 0.0098 + 0.0010 = 0.1719$.

(d) If the example were converted to a lower tail test, the p -value would have been the probability of obtaining 7 or fewer plus signs.

(e) The binomial probabilities provided in Table 5 of Appendix B can be used to compute the p -value when the sample size is 20 or less.

(f) With larger sample sizes, we rely on the normal distribution approximation of the binomial distribution to compute the p -value; this makes the computations quicker and easier.

Use the Normal Distribution to Approximate the Binomial Probability

1. **Example** One year ago the median price of a new home was \$236,000. However, a current downturn in the economy has real estate firms using sample data on recent home sales to determine if the population median price of a new home is less today than it was a year ago.

(a) The hypothesis test about the population median price of a new home is as follows:

$$H_0 : \text{Median} \geq 236,000$$

$$H_a : \text{Median} < 236,000$$

(b) We will use a 0.05 level of significance to conduct this test. A random sample of 61 recent new home sales found 22 homes sold for more than \$236,000, 38 homes sold for less than \$236,000, and one home sold for \$236,000.

- (c) After deleting the home that sold for the hypothesized median price of \$236,000, the sign test continues with 22 plus signs, 38 minus signs, and a sample of 60 homes.
- (d) The null hypothesis that the population median is greater than or equal to \$236,000 is expressed by the binomial distribution hypothesis $H_0 : p \geq 0.50$.
- (e) If H_0 were true as an equality, we would expect $0.50(60) = 30$ homes to have a plus sign.
- (f) The sample result showing 22 plus signs is in the lower tail of the binomial distribution. Thus, the p -value is the probability of 22 or fewer plus signs when $p = 0.50$.
- (g) While it is possible to compute the exact binomial probabilities for 0, 1, 2, \dots to 22 and sum these probabilities, we will use the normal distribution approximation of the binomial distribution to make this computation easier.

2. **Normal approximation of the sampling distribution of the number of plus signs when $H_0 : p = 0.50$:** For this approximation ($n > 20$), the mean and standard deviation of the normal distribution are:

$$\text{Mean : } \mu = \underline{np = 0.5n} \quad (18.1)$$

$$\text{Standard deviation : } \sigma = \underline{\sqrt{np(1-p)} = \sqrt{0.25n}} \quad (18.2)$$

3. With $n = 60$ homes and $p = 0.50$, the sampling distribution of the number of plus signs can be approximated by a normal distribution with

$$\mu = 0.50n = \underline{0.50(60) = 30}, \quad \sigma = \sqrt{0.25n} = \underline{\sqrt{0.25(60)} = 3.873}$$

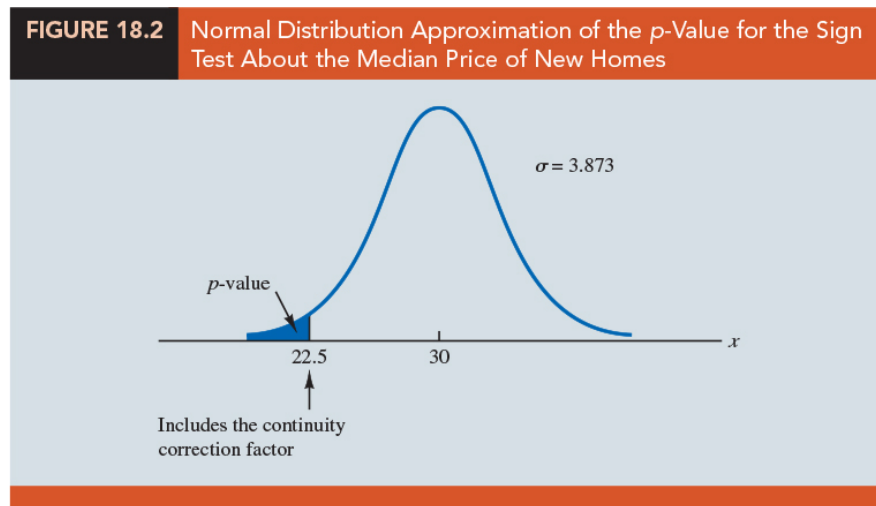
4. The binomial probability distribution is discrete and the normal probability distribution is continuous. To account for this, the binomial probability of 22 is computed by the normal probability interval 21.5 to 22.5. The 0.5 added to and subtracted from 22 is called the continuity correction factor.
5. Thus, to compute the p -value for 22 or fewer plus signs we use the normal distribution with $\mu = 30$ and $\sigma = 3.873$ to compute the probability that the normal random variable, X , has a value less than or equal to 22.5.

$$\underline{X \approx N(30, 3.873^2), \quad P(X \leq 22.5) = ?}$$

6. (Figure 18.2) Using this normal distribution, we compute the p -value as follows:

$$p\text{-value} = P(X \leq 22.5) = P\left(Z \leq \frac{22.5 - 30}{3.873}\right) = P(Z \leq -1.94) = 0.0262$$

7. With $0.0262 < 0.05$, we reject the null hypothesis and conclude that the median price of a new home is less than the \$236,000 median price a year ago.



Hypothesis Test with Matched Samples

1. (Recall Chapter 10) Using quantitative data and assuming that the differences between the pairs of matched observations were normally distributed, the t distribution was used to make an inference about the difference between the means of the two populations.
2. Use the nonparametric sign test to analyze matched-sample data. the sign test enables us to analyze categorical as well as quantitative data and requires no assumption about the distribution of the differences.
3. This type of matched-sample design occurs in market research when a sample of n potential customers is asked to compare two brands of a product such as coffee, soft drinks, or detergents. Without obtaining a quantitative measure of each individual's preference for the brands, each individual is asked to state a brand preference.

4. **Example** Sun Coast Farms produces an orange juice product called Citrus Valley. The primary competition for Citrus Valley comes from the producer of an orange juice known as Tropical Orange. In a consumer preference comparison of the two brands, 14 individuals were given unmarked samples of the two orange juice products. The brand each individual tasted first was selected randomly.
- If the individual selected Citrus Valley as the more preferred, a plus sign was recorded.
 - If the individual selected Tropical Orange as the more preferred, a minus sign was recorded.
 - If the individual was unable to express a difference in preference for the two products, no sign was recorded.
5. (Table 18.4) Deleting the two individuals who could not express a preference for either brand, the data have been converted to a sign test with 2 plus signs and 10 minus signs for the $n = 12$ individuals who could express a preference for one of the two brands.

TABLE 18.4 Preference Data for the Sun Coast Farms Taste Test

Individual	Preference	Sign	Individual	Preference	Sign
1	Tropical Orange	-	8	Tropical Orange	-
2	Tropical Orange	-	9	Tropical Orange	-
3	Citrus Valley	+	10	No Preference	
4	Tropical Orange	-	11	Tropical Orange	-
5	Tropical Orange	-	12	Citrus Valley	+
6	No Preference		13	Tropical Orange	-
7	Tropical Orange	-	14	Tropical Orange	-

6. Letting p indicate the proportion of the population of customers who prefer Citrus Valley orange juice, we want to test the hypotheses that there is no difference between the preferences for the two brands as follows:

$$H_0 : \underline{p = 0.50}$$

$$H_a : \underline{p \neq 0.50}$$

7. If H_0 cannot be rejected, we cannot conclude that there is a difference in preference for the two brands. However, if H_0 can be rejected, we can conclude that the consumer preferences differ for the two brands.

8. (Table 18.5) We will conduct the sign test ($\alpha = 0.05$). The sampling distribution for the number of plus signs is a binomial distribution with $p = 0.50$ and $n = 12$.
 ($X \sim B(n = 12, p = 0.5)$)

Number of Plus Signs	Probability
0	.0002
1	.0029
2	.0161
3	.0537
4	.1208
5	.1934
6	.2256
7	.1934
8	.1208
9	.0537
10	.0161
11	.0029
12	.0002

9. Under the assumption H_0 is true, we would expect $0.50n = 0.50(12) = 6$ plus signs. With only two plus signs in the sample, the results are in the lower tail of the binomial distribution.
10. To compute the p -value for this two-tailed test, we first compute the probability of 2 or fewer plus signs and then double this value. Using the binomial probabilities of 0, 1, and 2 shown in Table 18.5, the p -value is

$$p\text{-value} = \underline{2(0.0002 + 0.0029 + 0.0161) = 0.0384 < 0.05}$$

11. We reject H_0 . The taste test provides evidence that consumer preference differs significantly for the two brands of orange juice. We would advise Sun Coast Farms of this result and conclude that the competitor's Tropical Orange product is the more preferred. Sun Coast Farms can then pursue a strategy to address this issue.
12. Similar to other uses of the sign test, one-tailed tests may be used depending upon the application.
13. As the sample size becomes large, the normal distribution approximation of the binomial distribution will ease the computation.

14. While the Sun Coast Farms sign test for matched samples used categorical preference data, the sign test for matched samples can be used with quantitative data as well.
- (a) This would be particularly helpful if the paired differences are not normally distributed and are skewed.
 - (b) In this case a positive difference is assigned a plus sign, a negative difference is assigned a negative sign, and a zero difference is removed from the sample.
 - (c) The sign test computations proceed as before.

18.2 Wilcoxon Signed-Rank Test

1. (Recall Chapter 10) The parametric test for the matched-sample experiment requires quantitative data and the assumption that the differences between the paired observations are normally distributed. The t distribution can then be used to make an inference about the difference between the means of the two populations.
2. The Wilcoxon signed-rank test is a nonparametric procedure for analyzing data from a matched-sample experiment. The test uses quantitative data but does not require the assumption that the differences between the paired observations are normally distributed.
3. It only requires the assumption that the differences between the paired observations have a symmetric distribution.
4. This occurs whenever the shapes of the two populations are the same and the focus is on determining if there is a difference between the medians of the two populations.

5. **Example** Production Task Completion Times: Consider a manufacturing firm that is attempting to determine whether two production methods differ in terms of task completion time.

- (a) (Table 18.6) Using a matched-samples experimental design, 11 randomly selected workers completed the production task two times, once using method A and once using method B. The production method that the worker used first was randomly selected.

Worker	Method		Difference
	A	B	
1	10.2	9.5	.7
2	9.6	9.8	-.2
3	9.2	8.8	.4
4	10.6	10.1	.5
5	9.9	10.3	-.4
6	10.2	9.3	.9
7	10.6	10.5	.1
8	10.0	10.0	.0
9	11.2	10.6	.6
10	10.7	10.2	.5
11	10.6	9.8	.8

- (b) A positive difference indicates that method A required more time; a negative difference indicates that method B required more time.
- (c) Do the data indicate that the two production methods differ significantly in terms of completion times? If we assume that the differences have a symmetric distribution but not necessarily a normal distribution, the Wilcoxon signed-rank test applies.
- (d) In particular, we will use the Wilcoxon signed-rank test for the difference between the median completion times for the two production methods.

$$H_0 : \text{Median for method A} - \text{Median for method B} = 0$$

$$H_a : \text{Median for method A} - \text{Median for method B} \neq 0$$

- (e) If H_0 cannot be rejected, we will not be able to conclude that the median completion times are different. However, if H_0 is rejected, we will conclude that the median completion times are different.

6. The Wilcoxon signed-rank test steps ($\alpha = 0.05$):

- (a) Discard the difference of zero for worker 8 and then compute the absolute value of the differences for the remaining 10 workers.
- (b) Rank these absolute differences from lowest to highest. The first (second) smallest absolute difference of 0.1 (0.2) for worker 7 (2) is assigned the rank of 1 (2). This ranking of absolute differences continues with the largest absolute difference of 0.9 for worker 6 being assigned the rank of 10. The tied absolute differences of 0.4 (0.5) for workers 3 and 5 (4 and 10) are assigned the average rank of 3.5 (5.5).
- (c) (Table 18.7) Each rank is given the sign of the original difference for the worker.

Worker	Difference	Absolute Difference	Rank	Signed Ranks	
				Negative	Positive
1	.7	.7	8		8
2	-.2	.2	2	-2	
3	.4	.4	3.5		3.5
4	.5	.5	5.5		5.5
5	-.4	.4	3.5	-3.5	
6	.9	.9	10		10
7	.1	.1	1		1
8	.0				
9	.6	.6	7		7
10	.5	.5	5.5		5.5
11	.8	.8	9		9
				Sum of Positive Signed Ranks $T^+ = 49.5$	

- (d) Let T^+ denote the sum of the positive signed ranks ($T^+ = 49.5$). We will use T^+ as the Wilcoxon signed-rank test statistic.
- (e) **Sampling Distribution of T^+ for the Wilcoxon Signed-Rank Test:** If the medians of the two populations are equal and the number of matched pairs is 10 or more, the sampling distribution of T^+ can be approximated by a normal distribution :

$$\text{Mean : } \mu_{T^+} = \frac{n(n+1)}{4} \quad (18.3)$$

$$\text{Standard deviation : } \sigma_{T^+} = \sqrt{\frac{n(n+1)(2n+1)}{24}} \quad (18.4)$$

Distribution Form: Approximately normal for $n \geq 10$.

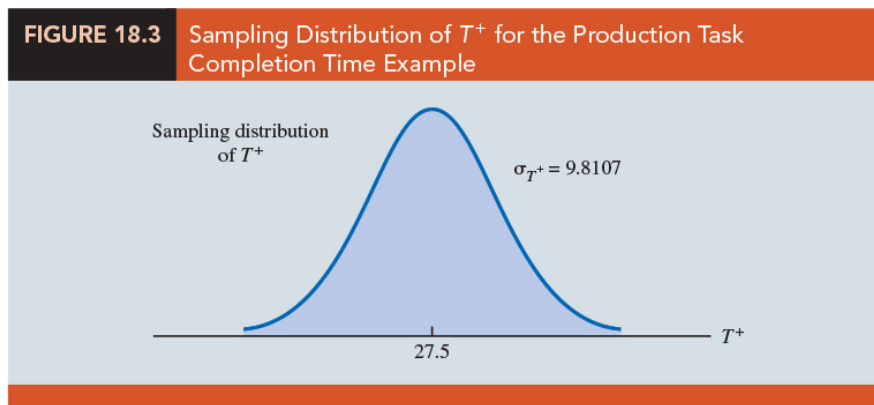
7. **Example** Production Task Completion Times:

- (a) After discarding the observation of a zero difference for worker 8, the analysis continues with the $n = 10$ matched pairs.

$$\mu_{T^+} = \frac{n(n+1)}{4} = \frac{10(10+1)}{4} = 27.5$$

$$\sigma_{T^+} = \sqrt{\frac{n(n+1)(2n+1)}{24}} = \sqrt{\frac{10(10+1)(10+1)}{24}} = \sqrt{\frac{2310}{24}} = 9.8107$$

- (b) (Figure 18.3) The sampling distribution of the T^+ test statistic.



- (c) Compute the two-tailed p -value for the hypothesis that the median completion times for the two production methods are equal. Since the test statistic $T^+ = 49.5$ is in the upper tail of the sampling distribution, we begin by computing the upper tail probability $P(T^+ \geq 49.5)$.
- (d) Since the sum of the positive ranks T^+ is discrete and the normal distribution is continuous, we will obtain the best approximation by including the continuity correction factor. Thus, the discrete probability of $T^+ = 49.5$ is approximated by the normal probability interval, 49 to 50, and the probability that $T^+ \geq 49.5$ is approximated by:

$$P(T^+ \geq 49.5) = P\left(z \geq \frac{49 - 27.5}{9.8107}\right) = P(z \geq 2.19)$$

(e) Using the standard normal distribution table and $z = 2.19$, we see that the two-tailed p -value = $2(1-0.9857) = 0.0286$. With the p -value ≤ 0.05 , we reject H_0 and conclude that the median completion times for the two production methods are not equal.

(f) With T^+ being in the upper tail of the sampling distribution, we see that method A led to the longer completion times. We would expect management to conclude that method B is the faster or better production method.

8. **One-tailed Wilcoxon signed-rank tests** are possible. For example, if initially we had been looking for statistical evidence to conclude method A had the larger median completion time than method B:

$$H_0 : \text{Median for method A} - \text{Median for method B} \leq 0$$

$$H_a : \text{Median for method A} - \text{Median for method B} > 0$$

9. (Recall Section 18.1) the sign test could be used for both a hypothesis test about a population median and a hypothesis test with matched samples.

10. The Wilcoxon signed-rank test can also be used for a nonparametric test about a population median. This test makes no assumption about the population distribution other than that it is symmetric.

11. If this symmetric assumption is appropriate, the Wilcoxon signed-rank test is the preferred nonparametric test for a population median. However, if the population is skewed, the sign test is preferred.

12. With the Wilcoxon signed-rank test, the differences between the observations and the hypothesized value of the population median are used instead of the differences between the matched-pair observations.

13. NOTES+COMMENTS:

(a) The Wilcoxon signed-rank test for a population median is based on the assumption that the population is symmetric. With this assumption, the population median is equal to the population mean. Thus, the Wilcoxon signed-rank test can also be used as a test about the mean of a symmetric population.

- (b) There are several variations of the Wilcoxon signed-rank test that generally provide similar but not identical results. The test we use in section 18.2 is based on a normal approximation (which is much easier to calculate).
- (c) JMP uses the exact Wilcoxon signed-rank test when $n \leq 20$ and a Student's t approximation when $n > 20$.

18.3 Mann-Whitney-Wilcoxon (MWW) Test

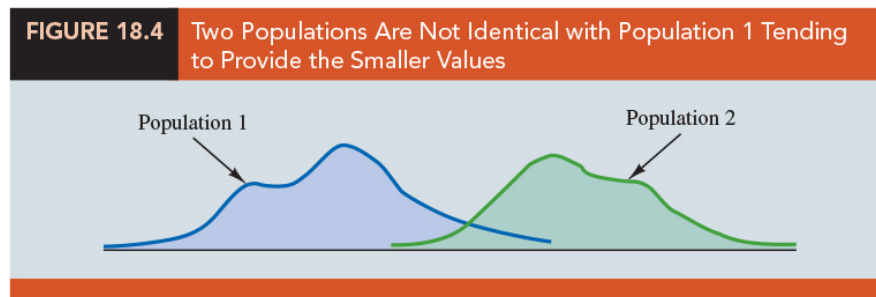
1. (Recall Chapter 10) A hypothesis test (t -test) about the difference between the means of two populations using two independent samples:
 - (a) This parametric test required quantitative data and the assumption that both populations had a normal distribution.
 - (b) If population standard deviations σ_1 and σ_2 were unknown, the sample standard deviations s_1 and s_2 provided estimates of σ_1 and σ_2 .
 - (c) The t distribution was used to make an inference about the difference between the means of the two populations.
2. We present a nonparametric test for the difference between two populations based on two independent samples. It can be used with either ordinal data or quantitative data and it does not require the assumption that the populations have a normal distribution.
3. Versions of the test were developed jointly by Mann and Whitney and also by Wilcoxon. As a result, the test has been referred to as the Mann-Whitney test and the Wilcoxon rank-sum test. The tests are equivalent and both versions provide the same conclusion. We will refer to this nonparametric test as the Mann-Whitney-Wilcoxon (MWW) test (e.g., a two-tailed test):

H_0 : The two populations are identical

H_a : The two populations are not identical

(i.e., either population may provide the smaller or larger values.)

4. If H_0 is rejected, we are using the test to conclude that the populations are not identical and that population 1 tends to provide either smaller or larger values than population 2.
5. (Figure 18.4) A situation where population 1 tends to provide smaller values than population 2. (Note that it is not necessary that all values from population 1 be less than all values from population 2.)



6. First illustrate the MWW test using small samples with rank-ordered data. Later, we will introduce a large-sample approximation based on the normal distribution that will simplify the calculations required by the MWW test.
7. **Example** Consider the on-the-job performance ratings for employees at a Showtime Cinemas 20-screen multiplex movie theater.
 - (a) During an employee performance review, the theater manager rated all 35 employees from best (rating 1) to worst (rating 35) in the theater's annual report. Knowing that the part-time employees were primarily college and high school students, the district manager asked if there was evidence of a significant difference in performance for college students compared to high school students.
 - (b) In terms of the population of college students and the population of high school students who could be considered for employment at the theater, the hypotheses were:

H_0 : College and high school student populations are identical
in terms of performance

H_a : College and high school student populations are not identical
in terms of performance

- (c) (Table 18.8) The theater manager's overall performance rating based on all 35 employees was recorded for each of these employees.

College Student	Manager's Performance Rating	High School Student	Manager's Performance Rating
1	15	1	18
2	3	2	20
3	23	3	32
4	8	4	9
		5	25

- (d) (Table 18.9)(**The combined-sample ranks**) Use a 0.05 level of significance for this test and rank the combined samples from low to high.

College Student	Manager's Performance Rating	Rank	High School Student	Manager's Performance Rating	Rank
1	15	4	1	18	5
2	3	1	2	20	6
3	23	7	3	32	9
4	8	2	4	9	3
	Sum of Ranks	14	5	25	8
				Sum of Ranks	31

- (e) **Sum the ranks** for each sample as shown in Table 18.9. The sum of ranks for the first sample will be the test statistic W for the MWW test: $W = 4 + 1 + 7 + 2 = 14$.
- (f) We will always follow the procedure of using the sum of the ranks for sample 1 as the test statistic W .
8. Why the sum of the ranks will help us select between the two hypotheses: H_0 : The two populations are identical and H_a : The two populations are not identical.

- (a) Letting C denote a college student and H denote a high school student, suppose the ranks of the nine students had the following order with the four college students having the four lowest ranks.

Rank	1	2	3	4	5	6	7	8	9
Student	C	C	C	C	H	H	H	H	H

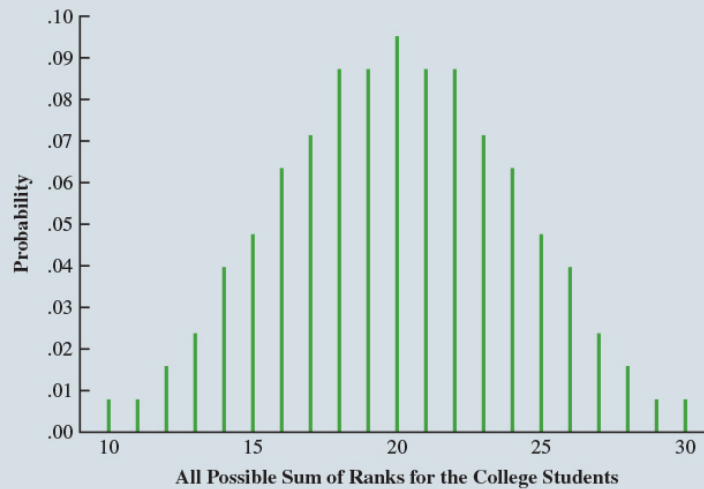
- (b) Notice that this permutation or ordering separates the two samples, with the college students all having a lower rank than the high school students.
- (c) This is a strong indication that the two populations are not identical. The sum of ranks for the college students in this case is $W = 1 + 2 + 3 + 4 = 10$.
- (d) Now consider a ranking where the four college students have the four highest ranks.

Rank	1	2	3	4	5	6	7	8	9
Student	H	H	H	H	H	C	C	C	C

This is another strong indication that the two populations are not identical.

The sum of ranks for the college students in this case is $W = 6 + 7 + 8 + 9 = 30$.

- (e) Thus, we see that the sum of the ranks for the college students must be between 10 and 30. Values of W near 10 imply that college students have lower ranks than the high school students, whereas values of W near 30 imply that college students have higher ranks than the high school students.
- (f) Either of these extremes would signal the two populations are not identical. However, if the two populations are identical, we would expect a mix in the ordering of the C 's and H 's so that the sum of ranks W is closer to the average of the two extremes, or nearer to $(10 + 30)/2 = 20$.
9. (Figure 18.5)(Table 18.10) Making the assumption that the two populations are identical, we used a computer program to compute all possible orderings for the nine students. For each ordering, we computed the sum of the ranks for the college students. This provided the probability distribution showing the exact sampling distribution of W .

FIGURE 18.5 Exact Sampling Distribution of the Sum of the Ranks for the Sample of College Students**TABLE 18.10** Probabilities for the Exact Sampling Distribution of the Sum of the Ranks for the Sample of College Students

W	Probability	W	Probability
10	.0079	20	.0952
11	.0079	21	.0873
12	.0159	22	.0873
13	.0238	23	.0714
14	.0397	24	.0635
15	.0476	25	.0476
16	.0635	26	.0397
17	.0714	27	.0238
18	.0873	28	.0159
19	.0873	29	.0079
		30	.0079

10. Use the sampling distribution of W in Figure 18.5 to compute the p -value for the test. Table 18.9 shows that the sum of ranks for the four college student is $W = 14$. Because this value of W is in the lower tail of the sampling distribution, we begin by computing the lower tail probability $P(W \leq 14)$:

$$\begin{aligned} P(W \leq 14) &= P(10) + P(11) + P(12) + P(13) + P(14) \\ &= 0.0079 + 0.0079 + 0.0159 + 0.0238 + 0.0397 = 0.0952 \end{aligned}$$

11. The two-tailed p -value $= 2(0.0952) = 0.1904$. With $\alpha = 0.05$ as the level of significance and p -value > 0.05 , the MWW test conclusion is that we cannot reject

the null hypothesis that the populations of college and high school students are identical.

12. Use the same combined-sample ranking procedure and use the normal distribution approximation of W to compute the p -value and draw the conclusion.
13. Example Third National Bank.
 - (a) The bank manager is monitoring the balances maintained in checking accounts at two branch banks and is wondering if the populations of account balances at the two branch banks are identical.
 - (b) (Table 18.11) Two independent samples of checking accounts are taken with sample sizes $n_1 = 12$ at branch 1 and $n_2 = 10$

Branch 1		Branch 2	
Account	Balance (\$)	Account	Balance (\$)
1	1095	1	885
2	955	2	850
3	1200	3	915
4	1195	4	950
5	925	5	800
6	950	6	750
7	805	7	865
8	945	8	1000
9	875	9	1050
10	1055	10	935
11	1025		
12	975		

- (c) (Table 18.12) The first step in the MWW test is to rank the combined data from the lowest to highest values. In that case of the two or more values are the same, the tied values are assigned the average rank of their positions in the combined data set.

TABLE 18.12 Assigned Ranks for the Combined Account Balance Samples

Branch	Account	Balance	Rank
2	6	750	1
2	5	800	2
1	7	805	3
2	2	850	4
2	7	865	5
1	9	875	6
2	1	885	7
2	3	915	8
1	5	925	9
2	10	935	10
1	8	945	11
1	6	950	12.5
2	4	950	12.5
1	2	955	14
1	12	975	15
2	8	1000	16
1	11	1025	17
2	9	1050	18
1	10	1055	19
1	1	1095	20
1	4	1195	21
1	3	1200	22

- (d) (Table 18.13) The next step is to sum the ranks for each sample: 169.5 for sample 1 and 83.5 for sample 2 are shown. Thus, we have $W = 169.5$. When both samples sizes are 7 or more, a normal approximation of the sampling distribution of W can be used.

TABLE 18.13 Combined Ranking of the Data in the Two Samples from Third National Bank

Branch 1			Branch 2		
Account	Balance (\$)	Rank	Account	Balance (\$)	Rank
1	1095	20	1	885	7
2	955	14	2	850	4
3	1200	22	3	915	8
4	1195	21	4	950	12.5
5	925	9	5	800	2
6	950	12.5	6	750	1
7	805	3	7	865	5
8	945	11	8	1000	16
9	875	6	9	1050	18
10	1055	19	10	935	10
11	1025	17			
12	975	15			
	Sum of Ranks	169.5		Sum of Ranks	83.5

14. Under the assumption that the null hypothesis is true and the populations are identical, the sampling distribution of the test statistic W is:

$$\text{Mean : } \underline{\mu_W = (1/2)n_1(n_1 + n_2 + 1)} \quad (18.5)$$

$$\text{Standard deviation : } \sigma_W = \sqrt{(1/12)n_1n_2(n_1 + n_2 + 1)} \quad (18.6)$$

Distribution form: Approximately normal provided $n_1 \geq 7$ and $n_2 \geq 7$.

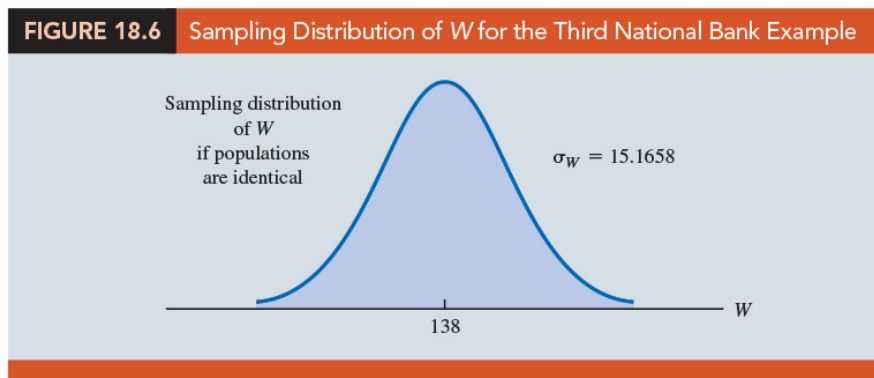
15. Since the test statistic W is discrete and the normal distribution is continuous, we will again use the continuity correction factor for the normal distribution approximation.

16. **Example** Third National Bank.

(a) (Figure 18.6) Given the sample sizes $n_1 = 12$ and $n_2 = 10$, equations (18.5) and (18.6) provide the following mean and standard deviation for the sampling distribution:

$$\text{Mean : } \mu_W = (1/2)(12)(12 + 10 + 1) = 138$$

$$\text{Standard deviation : } \sigma_W = \sqrt{(1/12)(12)(10)(12 + 10 + 1)} = 15.1658$$



(b) With $W = 169.5$ in the upper tail of the sampling distribution, we have the following p -value calculation:

$$P(W \geq 169.5) = P\left(z \geq \frac{169 - 138}{15.1658}\right) = P(z \geq 2.04)$$

(c) Using the standard normal random variable and $z = 2.04$, the two-tailed p -value $= 2(1 - 0.9793) = 0.0414$. With p -value ≤ 0.05 , reject H_0 and conclude that the two populations of account balances are not identical. The upper tail value for test statistic W indicates that the population of account balances at branch 1 tends to be larger.

17. Some applications of the MWW test make it appropriate to assume that the two populations have identical shapes and if the populations differ, it is only by a shift in the location of the distributions.
18. If the two populations have the same shape, the hypothesis test may be stated in terms of the difference between the two population medians. Any difference between the medians can be interpreted as the shift in location of one population compared to the other. In this case, the three forms of the MWW test about the medians ($M_i, i = 1, 2$) of the two populations are as follows:

Two-Tailed Test	Lower Tail Test	Upper Tail Test
$H_0: M_1 - M_2 = 0$	$H_0: M_1 - M_2 \geq 0$	$H_0: M_1 - M_2 \leq 0$
$H_a: M_1 - M_2 \neq 0$	$H_a: M_1 - M_2 < 0$	$H_a: M_1 - M_2 > 0$

18.4 Kruskal-Wallis Test

- (Recall Chapter 13, ANOVA) We considered a parametric test for three or more populations when we used quantitative data and assumed that the populations had normal distributions with the same standard deviations. Based on an independent random sample from each population, we used the F distribution to test for differences among the population means.
- The nonparametric Kruskal-Wallis test is based on the analysis of independent random samples from each of k populations. This procedure can be used with either ordinal data or quantitative data and does not require the assumption that the populations have normal distributions:

$$H_0 : \text{All populations are } \underline{\text{identical}}$$

$$H_a : \text{Not all populations are identical}$$

- If H_0 is rejected, we will conclude that there is a difference among the populations with one or more populations tending to provide smaller or larger values compared to the other populations.

4. **Example** Performance Evaluation Ratings for 20 Williams Employees

- (a) (Table 18.14) Williams Manufacturing Company hires employees for its management staff from three different colleges. Recently, the company's personnel director began reviewing the annual performance reports for the management staff in an attempt to determine whether there are differences in the performance ratings among the managers who graduated from the three colleges. The performance rating shown for each manager is recorded on a scale from 0 to 100, with 100 being the highest possible rating.

College A	College B	College C
25	60	50
70	20	70
60	30	60
85	15	80
95	40	90
90	35	70
80		75

- (b) Suppose we want to test whether the three populations of managers are identical in terms of performance ratings. We will use a 0.05 level of significance for the test.
- (c) (Table 18.15) The first step in the Kruskal-Wallis procedure is to rank the combined samples from lowest to highest values. Note that we assigned the average ranks to tied performance ratings of 60, 70, 80, and 90.

College A	Rank	College B	Rank	College C	Rank
25	3	60	9	50	7
70	12	20	2	70	12
60	9	30	4	60	9
85	17	15	1	80	15.5
95	20	40	6	90	18.5
90	18.5	35	5	70	12
80	15.5			75	14
Sum of Ranks	95	Sum of Ranks	27	Sum of Ranks	88

5. The Kruskal-Wallis test statistic:

$$H = \left[\frac{12}{n_T(n_T + 1)} \sum_{i=1}^k \frac{R_i^2}{n_i} \right] - 3(n_T + 1) \quad (18.7)$$

where

- k = the number of populations
 - n_i = the number of observations in sample i
 - $n_T = \sum_{i=1}^k n_i$ = the total number of observations in all samples
 - R_i = the sum of the ranks for sample i
- (a) Kruskal and Wallis were able to show that, under the null hypothesis assumption of identical populations, the sampling distribution of H can be approximated by a chi-square distribution with $(k-1)$ degrees of freedom.
- (b) This approximation is generally acceptable if the sample sizes for each of the k populations are all greater than or equal to five.
- (c) The null hypothesis of identical populations will be rejected if the test statistic H is large. As a result, the Kruskal-Wallis test is always expressed as an upper tail test.

6. Example Performance Evaluation Ratings for 20 Williams Employees

- (a) The value of the Kruskal-Wallis test statistic:

$$H = \frac{12}{20(21)} \left[\frac{(95)^2}{7} + \frac{(27)^2}{6} + \frac{(88)^2}{7} \right] - 3(20 + 1) = 8.92$$

- (b) We find $\chi_{0.025,2}^2 = 7.378$ has an area of 0.025 in the upper tail of the chi-square distribution and $\chi_{0.01,2}^2 = 9.21$ has an area of 0.01 in the upper tail of the chi-square distribution.
- (c) With $H = 8.92$ between 7.378 and 9.21, we can conclude that the p -value is between 0.025 and 0.01. Because $p\text{-value} \leq \alpha = 0.05$, we reject H_0 and conclude that the three populations are not all the same. The three populations of performance ratings are not identical and differ significantly depending upon the college.
- (d) Because the sum of the ranks is relatively low for the sample of managers who graduated from college B, it would be reasonable for the company to either reduce its recruiting from college B, or at least evaluate the college B graduates more thoroughly before making a hiring decision.

7. In some applications of the Kruskal-Wallis test, it may be appropriate to make the assumption that the populations have identical shapes and if they differ, it is only by a shift in location for one or more of the populations.
8. If the k populations are assumed to have the same shape, the hypothesis test can be stated in terms of the population medians. In this case, the hypotheses for the Kruskal-Wallis test would be written as follows:

$$H_0 : M_1 = M_2 = \cdots = M_k$$

$$H_a : \text{Not all Medians are equal}$$

9. NOTES+COMMENTS: The example in this section used quantitative data on employee performance ratings to conduct the Kruskal-Wallis test. This test could also have been used if the data were the ordinal rankings of the 20 employees in terms of performance. In this case, the test would use the ordinal data directly. The step of converting the quantitative data into rank-ordered data would not be necessary.

18.5 Rank Correlation

- (Recall Chapter 3) The Pearson product moment correlation coefficient is a measure of the linear association between two variables using quantitative data.
- The Spearman rank-correlation coefficient has been developed for a correlation measure of association between two variables when ordinal or rank-ordered data are available:

$$r_s = 1 - \frac{6 \sum_{i=1}^n d_i^2}{n(n^2 - 1)} \quad (18.8) \quad ()$$

where

n = the number of observations in the sample

x_i = the rank of observation i with respect to the first variable

y_i = the rank of observation i with respect to the second variable

$$d_i = x_i - y_i$$

3. The Spearman rank-correlation coefficient ranges from -1.0 to +1.0 and its interpretation is similar to the Pearson product moment correlation coefficient for quantitative data.

4. A rank-correlation coefficient near +1.0 (-1.0) (0) indicates a strong positive (negative) (no) association between the ranks for the two variables.

5. **Example** Sales Potential and Actual Two-Year Sales Data

(a) A company wants to determine whether individuals who had a greater potential at the time of employment turn out to have higher sales records. To investigate, the personnel director reviewed the original job interview reports, academic records, and letters of recommendation for 10 current members of the sales force.

(b) After the review, the director ranked the 10 individuals in terms of their potential for success at the time of employment and assigned the individual who had the most potential the rank of 1.

(c) (Table 18.16) Data were then collected on the actual sales for each individual during their first two years of employment. On the basis of the actual sales records, a second ranking of the 10 individuals based on sales performance was obtained.

Salesperson	Ranking of Potential	Two-Year Sales (units)	Ranking According to Two-Year Sales
A	2	400	1
B	4	360	3
C	7	300	5
D	1	295	6
E	6	280	7
F	3	350	4
G	10	200	10
H	9	260	8
I	8	220	9
J	5	385	2

- (d) (Table 18.17) Computation of the Spearman Rank-Correlation Coefficient for Sales Potential and Sales Performance

Salesperson	$x_i =$ Ranking of Potential	$y_i =$ Ranking of Sales Performance	$d_i = x_i - y_i$	d_i^2
A	2	1	1	1
B	4	3	1	1
C	7	5	2	4
D	1	6	-5	25
E	6	7	-1	1
F	3	4	-1	1
G	10	10	0	0
H	9	8	1	1
I	8	9	-1	1
J	5	2	3	9
				$\Sigma d_i^2 = 44$

$$r_s = 1 - \frac{6 \sum d_i^2}{n(n^2 + 1)} = 1 - \frac{6(44)}{10(100 - 1)} = .733$$

- (e) $r_s = 0.733$ indicates a positive correlation between the ranks based on potential and the ranks based on sales performance. Individuals who ranked higher in potential at the time of employment tended to rank higher in two-year sales performance.
6. Use the sample rank correlation r_s to make an inference about the population rank correlation coefficient ρ_s :

$$H_0 : \underline{\rho_s = 0} \quad H_a : \underline{\rho_s \neq 0}$$

7. (**S ampling distribution of r_s**) Under the assumption that the null hypothesis is true and the population rank-correlation coefficient is 0, the following sampling distribution of r_s can be used to conduct the test.

$$\text{Mean : } \underline{\mu_{r_s} = 0} \quad (18.9)$$

$$\text{Standard deviation : } \underline{\sigma_{r_s} = \sqrt{\frac{1}{n-1}}} \quad (18.10)$$

Distribution form: Approximately normal provided $n \geq 10$

8. Example Sales Potential and Actual Two-Year Sales Data

- (a) The sample rank-correlation coefficient for sales potential and sales performance is $r_s = 0.733$. Using equation (18.9), we have $\mu_{r_s} = 0$, and using equation (18.10), we have $\sigma_{r_s} = \sqrt{1/(10-1)} = 0.333$.
- (b) With the sampling distribution of r_s approximated by a normal distribution, the standard normal random variable z becomes the test statistic with

$$z = \frac{r_s - \mu_{r_s}}{\sigma_{r_s}} = \frac{0.733 - 0}{0.333} = 2.20$$

- (c) Using the standard normal probability table and $z = 2.20$, we find the two-tailed p -value $= 2(1 - .9861) = 0.0278$. With a 0.05 level of significance, $p\text{-value} \leq \alpha$. Thus, we **reject** the null hypothesis that the population rank-correlation coefficient is zero.
- (d) The test result shows that there is a **significant** rank correlation between potential at the time of employment and actual sales performance.

☺ EXERCISES

18.1 : 1, 3, 6, 9

18.2 : 12, 15, 17

18.3 : 18, 21

18.4 : 26, 29

18.5 : 32, 35

SUP : 39, 41, 45

“對一個不滿意的人生你只有兩種選擇，強迫自己接受，或說服自己改變。”

“You can only do one of two things to an unsatisfying life: force yourself to accept it, or convince yourself to change.”

— 媽的多重宇宙 (*Everything Everywhere All at Once*, 2022)

國立政治大學 111 學年度第 2 學期 小考 (1) 考試命題紙

考試科目：統計學 (二)

開課班別：統計學整合開課

命題教授：吳漢銘

考試日期：3 月 21 日 (二) 14:10-15:50

※准帶項目打「O」，否則打「×」

1. 需加發計算紙或答案紙請在試題內封袋備註。
2. 為環保節能減碳，試題一律採雙面印刷，如有特殊印製需求，請註記：

本試題共3頁，印刷份數：50 份

計算機	課本	筆記	字典	手機平板筆電
O	×	×	×	×

備註：注意事項要看!! (範圍: §ch10~11)

注意事項: (1) 答案卷請寫上科目、系級、學號及姓名。(2) 請按題號順序書寫。(3) 每一題號需置於答案卷最左邊。(4) 中英文作答皆可。(5) 建議用深色原子筆。(6) 需要計算過程。(7) 總分共 120 分。(8) 請複寫下列宣誓詞至答案卷第一頁最上頭。

0. (不寫扣 5 分)

本人姓名 重視誠信與榮譽，以認真負責的態度參與於本次實體考試，我將恪遵各項考試規則，不從事任何不正當或舞弊行為，若如違反誓言，願受校方給予的最嚴厲處罰，謹誓。

1. (15%, 5% each) 名詞解釋 (若有寫公式，不能只列出，需說明所使用符號的意思及公式代表的意義):

- (a) 顯著水準 (level of significance) (註: 以「假設檢定」為例)
- (b) 配對資料 (matched samples)
- (c) 卡方分佈 (chi-square distribution)(註: 不用寫出其機率密度函數)

2. (35%) 簡答題/問答題

- (a) (5%) 進行統計的假設檢定時，例如：兩母體平均數的檢定或單一母體變異數的檢定，不管是左尾檢定 (Left-tailed test)、右尾檢定 (Right-tailed test) 和雙尾檢定 (Two-tailed test)，為什麼「等號」都是寫在虛無假設? (禁止只寫「這樣才能進行假設檢定」，而不寫原由。)
- (b) (5%) 不管進行任何的假設檢定，為何只要 p -值小於顯著水準 (α)，就要拒絕 H_0 ? (禁止只寫「 $p < \alpha$ 代表 H_0 是不可能發生或是不對的」，而不寫原由。)
- (c) (5%) 試舉一個應用「Hypothesis Testing for Equality of Two Population Variances」的「政大校園生活情境」範例，請明確定義「母體為何? 變數為何?」，及說明要了解的「問題為何?」。同時說明，資料中的變數需滿足什麼條件才能合適地應用此統計方法。
- (d) (10%) 進行統計的假設檢定，有哪三種做決策方式 (亦即，現有資料是否有充份證據拒絕 H_0)?
- (e) (10%) 進行統計推論時，經常需有資料來自於常態分佈的前提假設。若資料不是來自常態分佈，一般實務上的建議 (Practical Advice) 為何? 請以「Hypothesis Tests About $\mu_1 - \mu_2$: σ_1 and σ_2 Unknown」為例進行說明。

國立政治大學 111 學年度第 2 學期 小考 (1) 考試命題紙

考試科目：統計學 (二)

開課班別：統計學整合開課

命題教授：吳漢銘

考試日期：3 月 21 日 (二) 14:10-15:50

※准帶項目打「O」· 否則打「×」

1. 需加發計算紙或答案紙請在試題內封袋備註。
2. 為環保節能減碳· 試題一律採雙面印刷· 如有特殊印製需求· 請註記：

本試題共3頁· 印刷份數：50 份

計算機	課本	筆記	字典	手機平板筆電
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備註：注意事項要看!! (範圍: §ch10~11)

O	×	×	×	×
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3. (25%) **Salaries of Recent College Graduates.** The Tippie College of Business obtained the following results on the salaries of a recent graduating class:

- Finance Majors: $n_1 = 110$, $\bar{x}_1 = \$48,537$, $s_1^2 = \$3.24e+08$.
- Business Analytics Majors: $n_2 = 30$, $\bar{x}_2 = \$55,317$, $s_2^2 = \$1e+08$.

- (a) (5%) Formulate hypothesis so that, if the null hypothesis is rejected, we can conclude that salaries for Finance majors are significantly lower than the salaries of Business Analytics majors. Use $\alpha = 0.05$.
- (b) (10%) What is the value of the test statistic? What is the decision?
- (c) (5%) What is the p -value? (依機率表格給出範圍即可)
- (d) (5%) What is your conclusion?

4. (25%) **Repair Costs as Automobiles Age.** In its 2016 Auto Reliability Survey, Consumer Reports asked subscribers to report their maintenance and repair costs. Most individuals are aware of the fact that the average annual repair cost for an automobile depends on the age of the automobile. A researcher is interested in finding out whether the variance of the annual repair costs also increases with the age of the automobile. A sample of 25 automobiles 2 years old showed a sample standard deviation for annual repair costs of \$100 and a sample of 26 automobiles 4 years old showed a sample standard deviation for annual repair costs of \$170.

- (a) (5%) State the null and alternative versions of the research hypothesis that the variance in annual repair costs is larger for the older automobiles.
- (b) (10%) At $\alpha = 0.01$ level of significance, what is your conclusion? (using the critical value approach)
- (c) (10%) What is the p -value? Discuss the reasonableness of your findings.

國立政治大學 111 學年度第 2 學期 小考 (1) 考試命題紙

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備註：注意事項要看!! (範圍: §ch10~11)

O	×	×	×	×
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5. (20%) 關於兩母體比例的信賴區間 (Interval Estimate of the Difference Between Two Population Proportions):

- (a) (5%) 請寫出 $(1 - \alpha)100\%$ confidence interval for $p_1 - p_2$ 。
- (b) (15%) 承上，請導出 (證明) 公式。

機率表

Upper tail probability p .									
i	p	z	$t_{(80)}$	$t_{(85)}$	$t_{(90)}$	$\chi^2_{(15)}$	$\chi^2_{(16)}$	$F_{(24,25)}$	$F_{(25,24)}$
1	0.200	0.8416	0.8461	0.8459	0.8456	19.3107	20.4651	1.4091	1.4134
2	0.100	1.2816	1.2922	1.2916	1.2910	22.3071	23.5418	1.6890	1.6960
3	0.050	1.6449	1.6641	1.6630	1.6620	24.9958	26.2962	1.9643	1.9750
4	0.025	1.9600	1.9901	1.9883	1.9867	27.4884	28.8454	2.2422	2.2574
5	0.010	2.3263	2.3739	2.3710	2.3685	30.5779	31.9999	2.6203	2.6430
6	0.005	2.5758	2.6387	2.6349	2.6316	32.8013	34.2672	2.9176	2.9472

公式

$$df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1-1} \left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2-1} \left(\frac{s_2^2}{n_2}\right)^2}$$

注意：1、考試求公平及公正，請同學務必自律，維護學校與學生之榮譽。

2、考試時不得有交談、窺視、夾帶、抄襲、傳遞、代考或其它作弊等舞弊行為，考畢務必交卷，不得攜卷出場，違者依考場規則議處。

國立政治大學 111 學年度第 2 學期 期中考 考試命題紙

考試科目：統計學 (二)

開課班別：統計學整合開課

命題教授：吳漢銘

考試日期：4 月 18 日 (二) 13:10-14:50

※准帶項目打「O」，否則打「×」

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2. 為環保節能減碳，試題一律採雙面印刷，如有特殊印製需求，請註記：

備註：注意事項要看!! (範圍：§10~§13)

O	×	×	×	×
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注意事項: (1) 答案卷請寫上科目、系級、學號及姓名。(2) 請按題號順序書寫。(3) 每一題號需置於答案卷最左邊。(4) 中英文作答皆可。(5) 可用鉛筆，建議用深色原子筆。(6) 需要計算過程。(7) 小數請計算至小數點以下 4 位。(8) 交回題目卷及答案卷。(9) 總分共 120 分。(8) 請複寫下列宣誓詞至答案卷第一頁最上頭或空白處 (不寫扣 10 分)。

本人姓名 重視誠信與榮譽，以認真負責的態度參與於本次實體考試，我將恪遵各項考試規則，不從事任何不正當或舞弊行為，若如違反誓言，願受校方給予的最嚴厲處罰，謹誓。

1. (15%; 5% each) 統計名詞解釋 (不能只列出公式，需說明所使用符號的意思及公式代表的意義):

- (a) 顯著水準 (level of significance) (註: 以「假設檢定」為例)
- (b) 配對資料 (matched samples)
- (c) 卡方分佈 (chi-square distribution)(註: 不用寫出其機率密度函數)

2. (20%; 5% each) 問答題 (A): Juicy Bun Burger 就是棒美式餐廳政大店的營銷團隊想要了解他們最受歡迎的三項木碗沙拉套餐 (培根凱薩嫩雞肉沙拉、女王煙燻肋排沙拉、挪威煙燻鮭魚沙拉)，在來店用餐客戶中的滿意程度是否相同，於是他們計畫聘用一位有修統計學的政大學生，來幫忙收集數據並進行統計分析。對的! 你就是這位學生。

- (a) 請說明此研究中的 (i) 母體為何? (ii) 如何收集資料 (樣本)? (iii) 如何紀錄資料 (亦即，定義要訪查的變數及其可能的值為何)?
- (b) 此研究中要使用哪一種統計方法較合適?
- (c) (承上) 使用此統計方法是否有什麼假設或需要注意的地方? (若無，則寫「無」)
- (d) 除了上述的方法，你可以根據所收集的資料，再幫店家進行哪些統計分析? (列出一項要分析的問題及所相對應要採用的統計方法即可。)

3. (25%; 5% each) 問答題 (B): 有關卡方獨立性檢定 (Chi-square Test For Independence):

- (a) 試舉一個在政大校園中應用卡方獨立性檢定的例子。
- (b) 說明此例中的母體、樣本、變數及變數的單位和可能值為何? (提示: 若變數是性別，則其可能值為「男性、女性、不告知」; 若變數是身高，則其可能值為介於 120CM 到 200CM 之正數。)
- (c) 應用此例，寫出此檢定的虛無假設 (H_0) 及擇一假設 (H_a)。
- (d) 使用卡方獨立性檢定是否有什麼假設或需要注意的地方? (若無，則寫「無」)
- (e) 若所收集的變數為連續型資料，該怎麼進行卡方獨立性檢定?

考試日期：4 月 18 日 (二) 13:10-14:50

※准帶項目打「O」，否則打「×」

1. 需加發計算紙或答案紙請在試題內封袋備註。
2. 為環保節能減碳，試題一律採雙面印刷，如有特殊印製需求，請註記：

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計算機	課本	筆記	字典	手機平板筆電
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備註：注意事項要看!! (範圍: §10~§13)

O	×	×	×	×
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4. (30%) **Assembly Methods (modified)**. Assembly Methods. Three different assembly methods have been proposed for a new product. A completely randomized experimental design was chosen to determine which assembly method results in the greatest number of parts produced per hour, and 30 workers were randomly selected and assigned to use one of the proposed methods. The number of units produced by each worker follows.

Method	Data	mean	sum	sd	SS ^a
A	97, 73, 93, 100, 73, 91, 100, 86, 92, 95	90	900	9.9	81882
B	93, 100, 93, 55, 77, 91, 85, 73, 90, 83	84	840	13.0	72076
C	99, 94, 87, 66, 59, 75, 84, 72, 88, 86	81	810	12.6	67048

- (a) (15%) Use these data and test to see whether the mean number of parts produced is the same with each method. Use $\alpha = 0.05$. Construct the ANOVA Table. (Note: identify the range of the p -value.)
 - (b) (10%) Use Fisher's LSD procedure to test for the equality of the means for methods B and C. Use a .05 level of significance.
 - (c) (5%) Use the Bonferroni adjustment to test for a significant difference between means for methods B and C. Assume that a maximum overall experimentwise error rate of 0.05 is desired.
5. (30%) **Daily High Temperatures**. Bob Feller, an Iowa farmer, has recorded the daily high temperatures during the same five-day stretch in May over the past five years. Bob is interested in whether this data suggests that the daily high temperature obeys a normal distribution. Use $\alpha = 0.01$ and conduct a goodness of fit test to see whether the following sample appears to have been selected from a normal probability distribution in the following steps. ($\bar{x} = 71$, $sd = 17$, $sum = 1775$, $SS = 132957$, Use 10 classes.)

55 86 94 58 55 95 55 52 69 95 90 65 87 50 56 55 57 98 58 79 92 62 59 88 65

^aSS: sum of square.

國立政治大學 111 學年度第 2 學期 期中考 考試命題紙

考試科目：統計學 (二)

開課班別：統計學整合開課

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※准帶項目打「O」，否則打「×」

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計算機	課本	筆記	字典	手機平板筆電
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備註：注意事項要看!! (範圍: §10~§13)

O	×	×	×	×
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(conti.)

(a) (5%) What are the values of (i) and (ii)?

Percentage	z	Temperatures
0.1	-1.28	(i)
0.2	-0.84	(ii)
0.3	-0.52	62.16
0.4	-0.25	66.75
0.5	0.00	71.00
0.6	0.25	75.25
0.7	0.52	79.84
0.8	0.84	85.28
0.9	1.28	92.76

(b) (10%) Compute the observed frequencies for the Temperature intervals.

(c) (15%) Compute the value of the test statistic and draw decision and conclusion.

機率表

Upper tail probability p .									
i	p	z	$t_{(27)}$	$t_{(28)}$	$t_{(29)}$	$\chi^2_{(7)}$	$\chi^2_{(9)}$	$F_{(2,27)}$	$F_{(3,27)}$
1	0.200	0.8416	0.8551	0.8546	0.8542	9.8032	12.2421	1.7093	1.6558
2	0.100	1.2816	1.3137	1.3125	1.3114	12.0170	14.6837	2.5106	2.2987
3	0.050	1.6449	1.7033	1.7011	1.6991	14.0671	16.9190	3.3541	2.9604
4	0.025	1.9600	2.0518	2.0484	2.0452	16.0128	19.0228	4.2421	3.6472
5	0.010	2.3263	2.4727	2.4671	2.4620	18.4753	21.6660	5.4881	4.6009
6	0.005	2.5758	2.7707	2.7633	2.7564	20.2777	23.5894	6.4885	5.3611

公式

• $SSTR = \sum_{i=1}^k n_j (\bar{x}_j - \bar{\bar{x}})^2$, $LSD = t_{\alpha/2} \sqrt{MSE \left(\frac{1}{n_i} + \frac{1}{n_j} \right)}$

注意：1、考試求公平及公正，請同學務必自律，維護學校與學生之榮譽。

2、考試時不得有交談、窺視、夾帶、抄襲、傳遞、代考或其它作弊等舞弊行為，考畢務必交卷，不得攜卷出場，違者依考場規則議處。

國立政治大學 111 學年度第 2 學期 小考 (2) 考試命題紙

考試科目：統計學 (二)

開課班別：統計學整合開課

命題教授：吳漢銘

考試日期：5 月 23 日 (二) 14:10-15:50

※准帶項目打「O」，否則打「×」

1. 需加發計算紙或答案紙請在試題內封袋備註。

本試題共3頁，印刷份數：50 份

計算機	課本	筆記	字典	手機平板筆電
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2. 為環保節能減碳，試題一律採雙面印刷，如有特殊印製需求，請註記：

備註：注意事項要看!! (範圍：§ch14~15.8)

O	×	×	×	×
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注意事項: (1) 答案卷請寫上科目、系級、學號及姓名。(2) 請按題號順序書寫。(3) 每一題號需置於答案卷最左邊。(4) 中英文作答皆可。(5) 建議用深色原子筆。(6) 需要計算過程。(7) 總分共 120 分。

0. (請複寫下列宣誓詞至答案卷第一頁最上頭或空白處。不寫扣 5 分)

本人姓名 重視誠信與榮譽，以認真負責的態度參與於本次實體考試，我將恪遵各項考試規則，不從事任何不正當或舞弊行為，若如違反誓言，願受校方給予的最嚴厲處罰，謹誓。

1. (50%) 簡答題/問答題

(a) (10%) 試寫出 (i) 簡單線性迴歸模型 (Simple Linear Regression Model, SLR) 及 (ii) 此模型的迴歸方程式。(需說明每一個數學符號的意思)。

(b) (10%) 為了在 SLR 中進行顯著性檢定 (tests of significance)，我們會對誤差項 ϵ 做什麼樣假設？

(c) (10%) (承上) 如何驗證模型的假設？(列舉課本裡提到的方法或工具，及說明如何利用這些方法或工具。)

(d) (10%) 進行 SLR 分析，如何檢測離群值 (Outliers) 及高影響點 (Influential Observations)? (列舉課本裡提到的方法或工具，及說明如何利用這些方法或工具。不需要寫出明確的公式。)

(e) (5%) 在簡單線性迴歸或多重迴歸分析中，變異數分析表格 (ANOVA Table) 為何是 F 檢定? (即為何檢定統計量是 F 分佈?)

(f) (5%) 多重迴歸分析中，為何需要計算「修正後的判定係數」？其計算公式為何？

2. (10%) 以最小平方法，導出簡單線性迴歸模型中迴歸係數 (β_0, β_1) 之估計式 (b_0, b_1) 。

3. (10%) 簡單線性迴歸模型中，請證明 $SST = SSR + SSE$ 。

國立政治大學 111 學年度第 2 學期 小考 (2) 考試命題紙

考試科目：統計學 (二)

開課班別：統計學整合開課

命題教授：吳漢銘

考試日期：5 月 23 日 (二) 14:10-15:50

※准帶項目打「O」，否則打「×」

1. 需加發計算紙或答案紙請在試題內封袋備註。
2. 為環保節能減碳，試題一律採雙面印刷，如有特殊印製需求，請註記：

本試題共3頁，印刷份數：50 份

計算機	課本	筆記	字典	手機平板筆電
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備註：注意事項要看!! (範圍：§ch14~15.8)

O	×	×	×	×
---	---	---	---	---

4. (30%) **Online Education.** One of the biggest changes in higher education in recent years has been the growth of online universities. The Online Education Database is an independent organization whose mission is to build a comprehensive list of the top accredited online colleges. The following table shows the retention rate (%) (就學穩定率或留校率) and the graduation rate (%) for 29 online colleges.

Retention Rate (%): 7, 51m 4, 29, 33, ..., 68, 100, 100

Graduation Rate (%): 25, 25, 28, 32, 33, ..., 56, 57, 61

- (a) (5%) Develop the estimated regression equation.
- (b) (5%) Test for a significant relationship. Use $\alpha = 0.05$. (需計算出 t 值。)
- (c) (5%) Did the estimated regression equation provide a good fit? (需完成 ANOVA 表)
- (d) (5%) 報表中，解釋 Coefficient "RetentionRate" 所代表的意義。
- (e) (10%) 計算 Adjusted R-squared, 並解釋。

```
> summary(OnlineEdu_lm)

Call:
lm(formula = GraduationRate ~ RetentionRate, data = OnlineEdu)

Residuals:
    Min       1Q   Median       3Q      Max
-14.9337  -6.4945   0.9448   4.8067  13.9198

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  25.42290   3.74628    6.786 2.74e-07 ***
RetentionRate  0.28453   0.06063    4.691 6.95e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 7.456 on 27 degrees of freedom
Multiple R-squared:  0.4492,    Adjusted R-squared:  0.4111
F-statistic: 22.02 on 1 and 27 DF,  p-value: 6.955e-05

> anova(OnlineEdu_lm)
Analysis of Variance Table

Response: GraduationRate
            Df Sum Sq Mean Sq F value    Pr(>F)
RetentionRate  1  1224.3    1224.3    22.02 6.955e-05 ***
Residuals    27  1501.0     55.59    1.00  <0.0001 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

考試日期：5 月 23 日 (二) 14:10-15:50

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計算機	課本	筆記	字典	手機平板筆電
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2. 為環保節能減碳，試題一律採雙面印刷，如有特殊印製需求，請註記：

備註：注意事項要看!! (範圍：§ch14~15.8)

O	×	×	×	×
---	---	---	---	---

5. (20%) **Risk of a Stroke.** A 10-year study conducted by the American Heart Association provided data on how age, blood pressure, and smoking relate to the risk of strokes. Assume that the following data are from a portion of this study. Risk is interpreted as the probability (times 100) that the patient will have a stroke over the next 10-year period. For the smoking variable, define a dummy variable with 1 indicating a smoker and 0 indicating a nonsmoker.

Risk: 12, 24, 13, ..., 34, 3, 37

Age: 57, 67, 58, ..., 80, 62, 59

Pressure: 152, 163, 155, ..., 125, 117, 207

Smoker: No, No, No, ..., Yes, No, Yes

- (a) (5%) Develop an estimated regression equation that relates risk of a stroke to the person's age, blood pressure, and whether the person is a smoker.
- (b) (5%) Is smoking a significant factor in the risk of a stroke? Explain. Use $\alpha = 0.05$.
- (c) (10%) What is the probability of a stroke over the next 10 years for Art Speen, a 68-year-old smoker who has blood pressure of 175? What action might the physician recommend for this patient?

```

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -91.75950   15.22276  -6.028 1.76e-05 ***
Age           1.07674    0.16596   6.488 7.49e-06 ***
Pressure      0.25181    0.04523   5.568 4.24e-05 ***
SmokerYes     8.73987    3.00082   2.912 0.0102 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 5.757 on 16 degrees of freedom
Multiple R-squared:  0.8735,    Adjusted R-squared:  0.8498
F-statistic: 36.82 on 3 and 16 DF,  p-value: 2.064e-07

> anova(Stroke_lm)
Analysis of Variance Table

Response: Risk
              Df Sum Sq Mean Sq F value    Pr(>F)
Age           1 1771.98 1771.98  53.4726 1.743e-06 ***
Pressure      1 1607.66 1607.66  48.5138 3.185e-06 ***
Smoker        1  281.10  281.10   8.4826 0.01017 *
Residuals    16  530.21   33.14
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
    
```

國立政治大學 111 學年度第 2 學期 期末考 考試命題紙

考試科目：統計學 (二)

開課班別：統計學整合開課

命題教授：吳漢銘

考試日期：6 月 13 日 (二) 13:10-14:50

※准帶項目打「O」，否則打「×」

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2. 為環保節能減碳，試題一律採雙面印刷，如有特殊印製需求，請註記：

本試題共4頁，印刷份數：50 份

計算機	課本	筆記	字典	手機平板筆電
O	×	×	×	×

備註：注意事項要看!! (範圍: §14~§18)

注意事項: (1) 答案卷請寫上科目、系級、學號及姓名。(2) 請按題號順序書寫。(3) 每一題號需置於答案卷最左邊。(4) 中英文作答皆可。(5) 可用鉛筆，建議用深色原子筆。(6) 需要計算過程。(7) 小數請計算至小數點以下 4 位。(8) 交回題目卷及答案卷。(9) 總分共 120 分。

0. (請複寫下列宣誓詞至答案卷第一頁最上頭或空白處。不寫扣 5 分)

本人姓名 重視誠信與榮譽，以認真負責的態度參與於本次實體考試，我將恪遵各項考試規則，不從事任何不正當或舞弊行為，若如違反誓言，願受校方給予的最嚴厲處罰，謹誓。

1. (20%; 5% each) 統計名詞解釋 (不能只列出公式，需說明所使用符號的意思及公式代表的意義):

- (a) Leverage (迴歸分析中的「槓桿」值)
- (b) Studentized deleted residual (學生化刪除殘差) (註: 需同時以文字說明是如何計算的)。
- (c) Odds ratio (指羅吉斯迴歸分析中的「勝算比」)
- (d) Nonparametric methods (無母數方法)

2. (20%) 羅吉斯迴歸模型中，當解釋變數個數只有 1 個的時候 (記為 X)，此模型稱為簡單羅吉斯迴歸模型 (Simple Logistic Regression Model):

- (a) (5%) 請寫出簡單羅吉斯迴歸模型的迴歸方程式 (Logistic Regression Equation)。
- (b) (5%) 若解釋變數的迴歸係數記為 β_1 ，其估計量記為 b_1 ，要如何解釋此「迴歸係數」?
- (c) (10%) 「勝算比」和迴歸係數的關係為何? 試證明之。

3. (15%) Multicollinearity

- (a) (5%) What does the multicollinearity mean in the multiple regression analysis?
- (b) (10%) What are the effects when the multicollinearity is severe (嚴重)? (請依課本內容作答)

4. (15%, 5% each) Wilcoxon Signed-Rank Test

- (a) 試舉一校園中的實例 (情境)，應用此檢定方法進行統計分析。(不可與考卷中的題目相同或類似)
- (b) 請寫出此實例中的「虛無假設」及「擇一假設」。(若有使用符號，需解釋其意義)
- (c) 要應用此檢定的前提 (或假設) 為何?

考試日期：6 月 13 日 (二) 13:10-14:50

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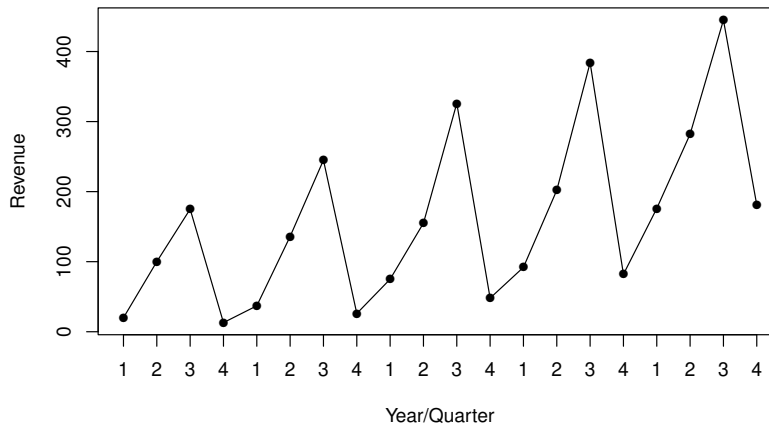
備註：注意事項要看!! (範圍：§14~§18)

O	×	×	×	×
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5. (15%) **Seasonal Sales.** South Shore Construction builds permanent docks and seawalls along the southern shore of Long Island, New York. Although the firm has been in business only five years, revenue has increased from \$308,000 in the first year of operation to \$1,084,000 in the most recent year. The following time series plot show the quarterly sales revenue in thousands of dollars. ((b)(c) 寫出模型，並說明符號意思)

- (a) (5%) What type of pattern exists in the data?
- (b) (5%) Develop a regression model to account for seasonal effects in the data.
- (c) (5%) Develop a regression model to account for seasonal effects and any linear trend in the time series.

SouthShore Time Series Plot



6. (15%) **Building Contracts.** The values of Alabama building contracts (in \$ millions) for a 12-month period follow: 240 350 230 260 280 320 220 310 240 310 240 230

- (a) (10%) Fill in the blanks in the following table where MA3 is the three-month moving average, ES0.2 is the exponential smoothing forecast using $\alpha = 0.2$. (計算空格的部份 (1)~(4) 即可，要有計算過程)(表中的 $|P.Error|$ 為 absolute value of percentage error.)
- (b) (5%) Which approach provides more accurate forecasts based on MSE?

國立政治大學 111 學年度第 2 學期 期末考 考試命題紙

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備註：注意事項要看!! (範圍: §14~§18)

O	×	×	×	×
---	---	---	---	---

t	Revenue	MA3	$ Error $	$(Error)^2$	$ P.Error $	ES0.2	$ Error $	$(Error)^2$	$ P.Error $
1	240								
2	350					240.00	110.00	12100.00	31.43
3	230	273.33	43.33	1877.78	18.84	262.00	32.00	1024.00	13.91
4	260	280.00	20.00	400.00	7.69	255.60	4.40	19.36	1.69
5	280	256.67	23.33	544.44	8.33	256.48	23.52	553.19	8.40
6	320	(1)	33.33	1111.11	(2)	(3)	58.82	3459.32	18.38
7	220	273.33	53.33	2844.44	24.24	(4)	52.95	2803.41	24.07
8	310	283.33	26.67	711.11	8.60	262.36	47.64	2269.78	15.37
9	240	256.67	16.67	277.78	6.94	271.89	31.89	1016.73	13.29
10	310	286.67	23.33	544.44	7.53	265.51	44.49	1979.45	14.35
11	240	263.33	23.33	544.44	9.72	274.41	34.41	1183.85	14.34
12	230	260.00	30.00	900.00	13.04	267.53	37.53	1408.18	16.32
Total	3230	2720.00	293.33	9755.56	115.36	2889.90	477.64	27817.28	171.54

7. (20%, 5% each) **Prices of Brands of Refrigerators.** Twelve homemakers were asked to estimate the retail selling price of two models of refrigerators. Their estimates of selling price are shown in the following table. Use these data and test at the 0.05 level of significance to determine whether there is a difference between the two models in terms of homemakers' perceptions of selling price.

- (a) Which statistical test can be used here? What is the H_0 ?
- (b) What is the value of the test statistic?
- (c) Using normal approximation and the continuity correction to compute the p -value.
- (d) Draw a conclusion after making the decision.

Homemaker	Model 1	Model 2	Difference
1	850	1100	-250
2	960	920	40
3	940	890	50
4	900	1050	-150
5	790	1120	-330
6	820	1000	-180
7	900	1090	-190
8	890	1120	-230
9	1100	1200	-100
10	700	890	-190
11	810	900	-90
12	920	900	20

國立政治大學 111 學年度第 2 學期 期末考 考試命題紙

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備註：注意事項要看!! (範圍：§14~§18)

O	×	×	×	×
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機率表

Upper tail probability p .

i	p	z	$t_{(27)}$	$t_{(28)}$	$t_{(29)}$	$\chi_{(7)}^2$	$\chi_{(9)}^2$	$F_{(2,27)}$	$F_{(3,27)}$
1	0.200	0.8416	0.8551	0.8546	0.8542	9.8032	12.2421	1.7093	1.6558
2	0.100	1.2816	1.3137	1.3125	1.3114	12.0170	14.6837	2.5106	2.2987
3	0.050	1.6449	1.7033	1.7011	1.6991	14.0671	16.9190	3.3541	2.9604
4	0.025	1.9600	2.0518	2.0484	2.0452	16.0128	19.0228	4.2421	3.6472
5	0.010	2.3263	2.4727	2.4671	2.4620	18.4753	21.6660	5.4881	4.6009
6	0.005	2.5758	2.7707	2.7633	2.7564	20.2777	23.5894	6.4885	5.3611

pnorm

pnorm(2.25) = 0.9877755, pnorm(2.35) = 0.9906133, pnorm(2.45) = 0.9928572, pnorm(2.55) = 0.9946139, pnorm(2.65) = 0.9959754, pnorm(2.75) = 0.9970202.

公式

- Mean: $\mu = np = 0.5n$, Standard deviation: $\sigma = \sqrt{np(1-p)} = \sqrt{0.25n}$
- Mean: $\mu_{T+} = \frac{n(n+1)}{4}$, Standard deviation: $\sigma_{T+} = \sqrt{\frac{n(n+1)(2n+1)}{24}}$, Distribution Form: Approximately normal for $n \geq 10$.
- Mean: $\mu_W = (1/2)n_1(n_1 + n_2 + 1)$, Standard deviation: $\sigma_W = \sqrt{(1/12)n_1n_2(n_1 + n_2 + 1)}$ Distribution form: Approximately normal provided $n_1 \geq 7$ and $n_2 \geq 7$.
- $h_i = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum(x_i - \bar{x})^2}$
- $\frac{y_i - \hat{y}_i}{s_{y_i - \hat{y}_i}}, s_{y_i - \hat{y}_i} = s\sqrt{1 - h_i}$
- $D_i = \frac{(y_i - \hat{y}_i)^2}{(p+1)s^2} \left[\frac{h_i}{(1 - h_i)^2} \right]$

注意：1、考試求公平及公正，請同學務必自律，維護學校與學生之榮譽。

2、考試時不得有交談、窺視、夾帶、抄襲、傳遞、代考或其它作弊等舞弊行為，考畢務必交卷，不得攜卷出場，違者依考場規則議處。

“不管你再怎麼努力，還是有人會忽略你的付出；就為了自己而奮鬥吧。”

“Your efforts will always be neglected no matter how hard you try; so fight for yourself.”

— 緊急迫降 (*Emergency Declaration, 2022*)