微 積 分

THOMAS′ CALCULUS 12/E

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開課班級: 資訊 1/電機 1/智財學程

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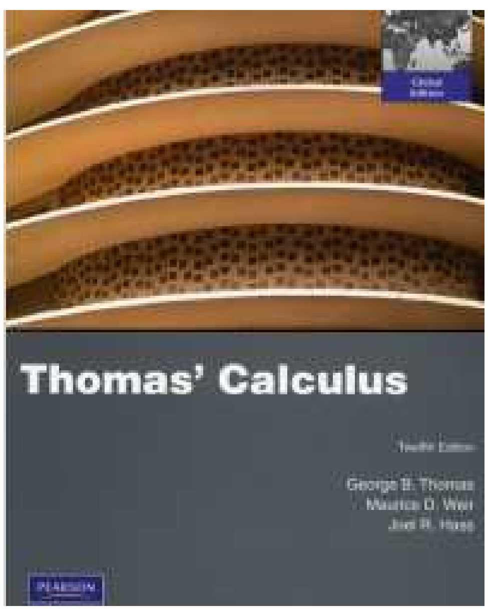

107 學年度第 1 學期

目錄

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叮嚀

A. 教不完是正常; 考不好是日常; 平常就要唸書!

B. 上課以「互相尊重」為最高原則並盡到「告知老師」的義務。

C. 上課可小聲討論、可上廁所安靜去回、可飲食。(但請一定要維護教室整潔)

D. 四不一要: 「上課不聊天，睡覺不趴著，手機不要滑，考試不作弊，要認真。」

THOMAS' CALCULUS (12/E)

2.1 Rates of Change and Tangents to Curves

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1 Average and Instantaneous Speed

1.1 **Average speed** during an interval of time: divide the ____________ covered by the _________ elapsed.

$$\frac{\Delta y}{\Delta t} = \underline{\hspace{4cm}}$$

1.2 **Average speed** over a time interval $[t_0, t_0 + h]$ having length ___________:

$$\frac{\Delta y}{\Delta t} = \underline{\hspace{5cm}}$$

1.3 The unit of measure is length per unit time; e.g., km/hr, m/sec.

✎ Ex. **1** .... (example1, p40)

A rock breaks loose from the top of a tall cliff. What is its average speed (a) during the first 2 sec of fall? (b) during the 1-sec interval between second 1 and second 2.

sol:

✎ Ex. **2** .... (`example2, p40`)

Find the speed of the falling rock in Ex. 1 at $t = 1$ and $t = 2$ sec.

sol:

2 Average Rates of Change and Secant Lines

2.1 *Definition: Average Rate of Change over an Interval*

> The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is
>
> $$\frac{\Delta y}{\Delta x} = \underline{\hspace{4cm}} = \underline{\hspace{5cm}}, \quad h \neq 0$$

2.2 The rate of change of f over $[x_1, x_2]$ is the __________________ through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$.

2.3 The slope of a **secant** to the graph $y = f(x)$ is ___________ (i.e., the average rate of change of f over the interval $[x_1, x_2]$). (如下圖)

Ex. 3 .... (example3, p42)

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

sol:

3 Instantaneous Rates of Change and Tangent Lines

3.1 **Instantaneous speed** at t_0 = average speed during ______________ as ___________________.

3.2 The __________________________ corresponds to the slope of a _______________;

3.3 The ______________________ corresponds to the slope of the ________________.

3.4 The limiting values of average rates is the _________________________________ (the rate at $t = t_0$).

實習課練習 (EXERCISE 2.1)

4. Find the average rate of change of the function g over the given intervals.
$g(t) = 2 + \cos t$; (a) $[0, \pi]$ (b) $[-\pi, \pi]$

5. Find the average rate of change of the function $R(\theta) = \sqrt{4\theta + 1}$ over the given intervals $[0, 2]$.

12. Find the slope of the curve $y = 2 - x^3$ at the given point $P(1, 1)$, and an equation of the tangent line at $P(1, 1)$.

19. Let $g(x) = \sqrt{x}$ for $x \geq 0$. (a) Find the average rate of change of $g(x)$ with respect to x over the intervals [1,2], [1, 1.5], and $[1, 1 + h]$. (b) Calculate the limit as h approaches zero.

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2.2 Limit of a Function and Limit Laws

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1 Limits of Function Values

1.1 Let $f(x)$ be defined on an ______________________, except possibly at x_0 itself. If $f(x)$ get close to L for all x sufficiently close to x_0, we say that f approaches the ________ L as x approaches x_0, and we write

_______________,

read as ____________________________________.

✎ Ex. **1** .... (`example1, p46`)

How does the function $f(x) = \dfrac{x^2-1}{x-1}$ behave near $x = 1$?

sol:

✎ Ex. **2** .... (`example3, p48`)

(a) If f is the **identity function** ___________, then for any value of x_0

$$\lim_{x\to x_0} f(x) = \_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_$$

(b) If f is the **constant function** ___________, then for any value of x_0

$$\lim_{x\to x_0} f(x) = \_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_$$

Ex. 3 .... (example4, p48)

A function may fail to have a limit at a point in its domain.

Discuss the behavior of the following functions as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$

(b) $g(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \leq 0, \\ \sin \dfrac{1}{x}, & x > 0. \end{cases}$

sol:

2 The Limit Laws

2.1 *Theorem 1: Limit Laws*

If L, M, c and k are ________ and ________ and ________, then

(a) sum rule: ________.

(b) difference rule: ________.

(c) constant multiple rule: ________.

(d) product rule: ________.

(e) quotient rule: ________.

(f) power rule: if r and s are integers with no commom factor and $s \neq 0$, then ________, provided that $L^{r/s}$ is a real number.

(g) root rule: ________, $n \in N^+$.

2.2 *Theorem 2: Limits of Polynomials*

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then
$$\lim_{x\to c} P(x) = \underline{\qquad\qquad} = \underline{\qquad\qquad\qquad\qquad\qquad\qquad}.$$

2.3 *Theorem 3: Limits of Rational Functions*

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then
$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \underline{\qquad\qquad}.$$

Ex. 4 .... (example5, p50)

Find the following limits:
(a) $\lim_{x\to c}(x^3 + 4x^2 - 3)$, (b) $\lim_{x\to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$, (c) $\lim_{x\to -2} \sqrt{4x^2 - 3}$.

sol:

3 Eliminating Zero Denominators Algebraically

3.1 Canceling a __________________.

3.2 Creating and canceling a __________________.

Ex. 5 .... (example7, p51)

Evaluate $\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - x}$.

sol:

Ex. 6 .... (example9, p52)

Evaluate $\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$.

sol:

4 The Sandwich Theorem

4.1 *Theorem 4: The Sandwich Theorem (The Squeeze Theorem, The Pinching Theorem)*

> Suppose that ________________ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that
> ____________________, then ____________.

4.2 (a) $\lim_{\theta\to 0} \sin\theta =$_____. (b) $\lim_{\theta\to 0} \cos\theta =$_____.

Ex. 7 .... (example10, p53)

Given that $1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2}$, for all $x \ne 0$, find $\lim_{x\to 0} u(x)$.

sol:

Ex. 8 .... (example11, p53)

For any function f, $\lim_{x\to c} |f(x)| = 0$ implies $\lim_{x\to c} f(x) = 0$.

sol:

實習課練習 (EXERCISE 2.2)

17. Find the limit: $\lim_{x \to -1} 3(2x-1)^2$.

22. Find the limit: $\lim_{h \to 0} \dfrac{\sqrt{5h+4}-2}{h}$.

36. Find the limit: $\lim_{x \to 4} \dfrac{4x - x^2}{2 - \sqrt{x}}$.

41. Find the limit: $\lim_{x \to -3} \dfrac{2 - \sqrt{x^2-5}}{x+3}$.

47. Find the limit: $\lim_{x \to 0} \dfrac{1 + x + \sin x}{3 \cos x}$.

63. If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x \to 0} f(x)$.

78. If $\lim_{x \to -2} \dfrac{f(x)}{x^2} = 1$, find (a) $\lim_{x \to -2} f(x)$, (b) $\lim_{x \to -2} \dfrac{f(x)}{x}$.

79. (a) If $\lim_{x \to 2} \dfrac{f(x)-5}{x-2} = 3$, find $\lim_{x \to 2} f(x)$.
(b) If $\lim_{x \to 2} \dfrac{f(x)-5}{x-2} = 4$, find $\lim_{x \to 2} f(x)$.

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2.3 The Precise Definition of a Limit

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1 Definition of Limit

1.1 *Definition: Limit of a Function*

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that ______________________________________, and we write

____

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

________________ $\Rightarrow$ ________________

1.2 說明圖示如下:

Ex. 1 .... (example2, p59)

Show that $\lim_{x \to 1}(5x - 3) = 2$.

sol:

Ex. 2 .... (example3, p60)

Prove (a) $\lim_{x \to x_0} x = x_0$, (b) $\lim_{x \to x_0} k = k$.

sol:

2 Find δ Algebraically for a Given f, L, x_0 and $\epsilon > 0$

Finding a ____

2.1 *Solve the inequality* ________________ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.

2.2 *Find a value of* _________ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b)

2.3 The inequality ________________ will hold for all $x \neq x_0$ in this δ-interval.

✎ Ex. **3** .... (example4, p60)

Show that $\lim_{x\to 5} \sqrt{x-1} = 2$.

sol:

✎ Ex. **4** .... (example5, p61)

Prove that $\lim_{x\to 2} f(x) = 4$ if $f(x) = \begin{cases} x^2, & x \neq 2. \\ 1, & x = 2. \end{cases}$

sol:

實習課練習 (EXERCISE 2.3)

40. Prove the limit: $\lim_{x\to 0}\sqrt{4-x}=2$.

43. Prove the limit: $\lim_{x\to 1}\frac{1}{x}=1$.

48. Prove that $\lim_{x\to 0} f(x)=0$ if $f(x)=\begin{cases} 2x, & x<0. \\ x/2, & x\geq 0. \end{cases}$

49. Prove the limit: $\lim_{x\to 0} x\sin\frac{1}{x}=0$.

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2.4 One-Sided Limits

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1 One-Sided Limits

1.1 Two-sided limits: ______________________________ from ____________.

1.2 Right-hand limit: ______________.

("__________": consider only values of x greater than c.)

1.3 Left-hand limit: ______________.

1.4 Example: $f(x) = \frac{x}{|x|}$, $\lim_{x\to 0^+} f(x) =$_____, and $\lim_{x\to 0^-} f(x) =$______.

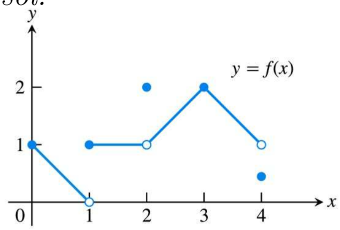
1.5 *Theorem 6*

> A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits and these one-sided limits are equal:
>
> ______________ $\Leftrightarrow$ ______________ and ______________

Ex. 1 .... (`example2, p68`)

Find limits of the function at $x = 0, 1, \cdots, 4$ graphed below.

sol:



2 Precise Definitions of One-Sided Limits

2.1 *Definitions: Right-Hand, Left-Hand Limits*

(a) We say that $f(x)$ has **right-hand limit L at** x_0, and write

____

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

____

(b) We say that $f(x)$ has **left-hand limit L at** x_0, and write

____

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

____

2.2 $f(x) = \sin(1/x)$ has ___________ as x approaches zero from either side.

3 Limits Involving $\dfrac{\sin\theta}{\theta}$

3.1 *Theorem 7*

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Proof:

✎ Ex. **2** .... (example5, p70)

Evaluate (a) $\lim_{h\to 0}\dfrac{\cos h-1}{h}$, (b) $\lim_{x\to 0}\dfrac{\sin 2x}{5x}$.

sol:

✎ Ex. **3** .... (example5, p70)

Find $\lim_{t\to 0}\dfrac{\tan t\sec 2t}{3t}$.

sol:

實習課練習

3. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2. \end{cases}$

(a) Find $\lim\limits_{x \to 2^+} f(x)$ and $\lim\limits_{x \to 2^-} f(x)$.

(b) Does $\lim\limits_{x \to 2} f(x)$ exist? If so, what is it? If not, why not?

(c) Find $\lim\limits_{x \to 4^-} f(x)$ and $\lim\limits_{x \to 4^+} f(x)$.

(d) Does $\lim\limits_{x \to 4} f(x)$ exist? If so, what is it? If not, why not?

5. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin \dfrac{1}{x}, & x > 0. \end{cases}$

(a) Does $\lim\limits_{x \to 0^+} f(x)$ exist? If so, what is it? If not, why not?

(b) Does $\lim\limits_{x \to 0^-} f(x)$ exist? If so, what is it? If not, why not?

(c) Does $\lim\limits_{x \to 0} f(x)$ exist? If so, what is it? If not, why not?

11. Find the limit: $\lim\limits_{x \to -0.5^-} \sqrt{\dfrac{x+2}{x+1}}$.

15. Find the limit: $\lim\limits_{h \to 0^+} \dfrac{\sqrt{h^2 + 4h + 5} - \sqrt{5}}{h}$.

18. Find the limit: (a) $\lim\limits_{x \to 1^+} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$, (b) $\lim\limits_{x \to 1^-} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$.

19. Find the limit: (a) $\lim\limits_{\theta \to 3^+} \dfrac{\lfloor \theta \rfloor}{\theta}$, (b) $\lim\limits_{\theta \to 3^-} \dfrac{\lfloor \theta \rfloor}{\theta}$. (note: $\lfloor x \rfloor$ is the greatest integer of x)

20. Find the limit: (a) $\lim\limits_{t \to 4^+} (t - \lfloor t \rfloor)$, (b) $\lim\limits_{t \to 4^-} (t - \lfloor t \rfloor)$. (note: $\lfloor x \rfloor$ is the greatest integer of x)

25. Find the limit: $\lim\limits_{x \to 0} \dfrac{\tan 2x}{x}$.

29. $\lim\limits_{x \to 0} \dfrac{x + x \cos x}{\sin x \cos x}$.

39. $\lim\limits_{x \to 0} \dfrac{\tan 3x}{\sin 8x}$.

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2.5 Continuity

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1 Continuity at a Point

1.1 *Definition: Continuity at a Point*

Interior point: A function $y = f(x)$ is continuous at an ______________ c of its domain if

$$\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_.$$

Endpoint: A function $y = f(x)$ is continuous at a ______________ a or is continuous at a ______________ b of its domain if

$$\_\_\_\_\_\_\_\_\_\_ \text{ or } \_\_\_\_\_\_\_\_\_\_.$$

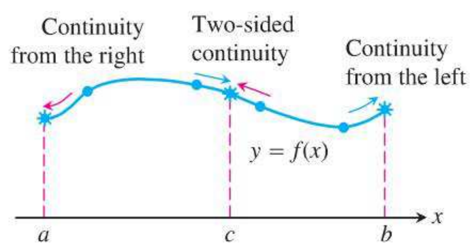


FIGURE 2.36 Continuity at points a, b, and c.

1.2 If a function is **not** continuous at a point c: f is ______________ at c, and c is a point of ______________ of f.

1.3 f is right-continuous (continuous from the right) at a point $x = c$ in its domain if

$$\_\_\_\_\_\_\_\_\_\_.$$

1.4 f is left-continuous (continuous from the left) at a point $x = c$ in its domain if

$$\underline{\hspace{4cm}}.$$

1.5 f is continuous at a **left** endpoint a of its domain if it is ________________ at a.

1.6 f is continuous at a **right** endpoint b of its domain if it is ________________ at b.

1.7 f is continuous at an **interior** point c of its domain if and only if it is both ________________ and ________________ at c.

1.8 *Continuity Test*

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

(a) ____________ : c lies in the domain of f.

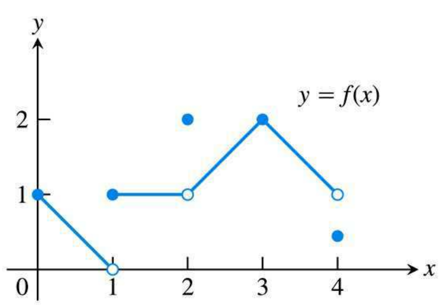
(b) ________________ : f has a limit as $x \to c$.

(c) ________________ : the limit equals the function value.

(a) To define continuity at a point in a function's domain, we need to define continuity at an ______________ (which involves a two-sided limit) and continuity at an ___________ (which involves a one-sided limit).

✎ **Ex. 1** .... (example1, p74)

Find the points at which the function f is continuous and the points at which f is discontinuous. (圖形如下)



sol:

✏ Ex. **2** .... (`example4, p75`)

The function $y = \lfloor x \rfloor$ is discontinuous at every integer because the left-hand and right-hand limits are not equal as $x \to n$:

$$\lim_{x \to n^-} \lfloor x \rfloor = \_\_\_\_\_\_\_\_\_ \qquad \lim_{x \to n^+} \lfloor x \rfloor = \_\_\_\_\_\_$$

Since $\lfloor n \rfloor = n$, the greatest integer function is right-continuous at every integer n.

2 Continuous Function

2.1 A function is continuous on an interval if and only if it is ___________________________ of the interval.

2.2 A ______________________ is one that is continuous at every point of its domain.

2.3 $y = 1/x$ is not continuous on $[-1, 1]$, but it is continuous over its domain ____________________.

2.4 *Theorem 8: Properties of Continuous Functions*

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

(a) *Sums* : __________

Prove the sum property:

(b) *Differences* : __________

(c) *Constant multiples* : _________

(d) *Products* : ________

(e) *Quotients* : ________________________

(f) *Powers* : ________ provided it is defined on an open interval, r, s: integers

(g) *Roots* : ________, provided it is defined on an open interval containing c, where n is a positive integer.

2.5 Every polynomial $P(x) =$ ____________________________ is continuous because ____________________.

2.6 *Theorem 9: Composite of Continuous Functions*

If f is continuous at c and g is continuous at $f(c)$, then the composite ________ is continuous at c.

2.7 *Theorem 10: Composite of Continuous Functions*

If g is continuous at b and $\lim_{x\to c} f(x) = b$, then

$$\lim_{x\to c} g(f(x)) = \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

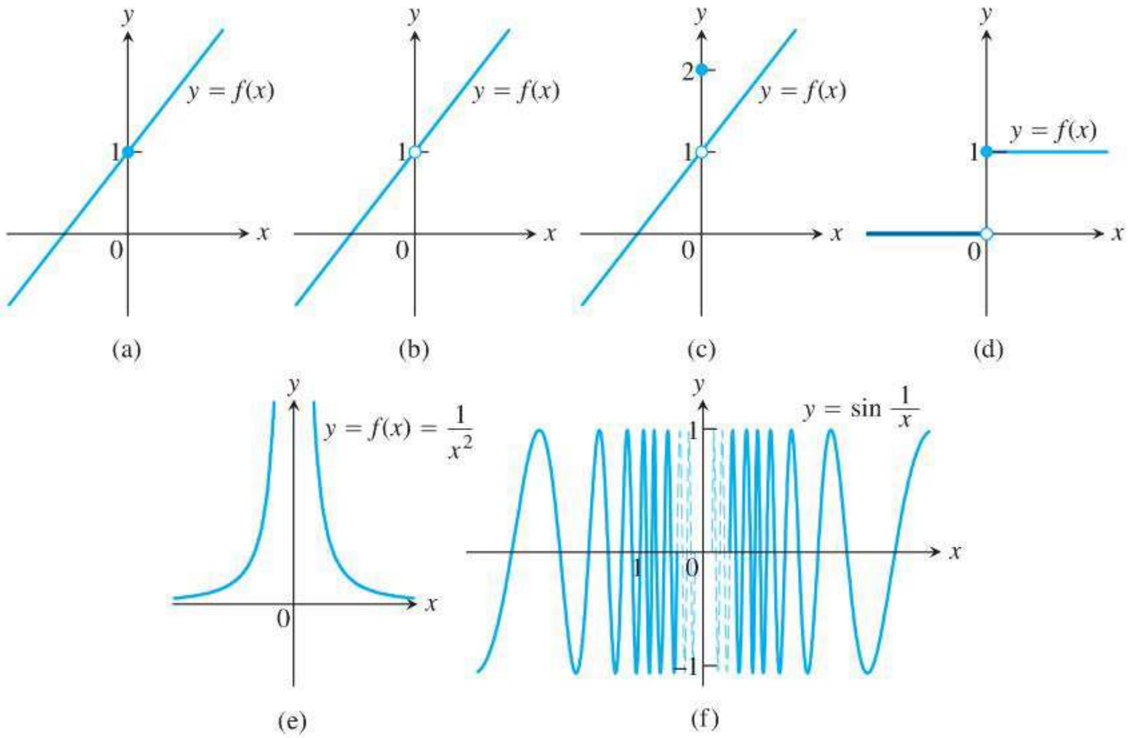


FIGURE 2.40 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

3 Continuous Extension to a Point

3.1 A function may have a limit even at a point where it is ______________.

3.2 If $f(c)$ is not defined, but __________________ exists, we can define a new function

$F(x)$ by the rule

____

3.3 The function F is continuous at $x = c$. It is called the ______________________ of f to _________.

3.4 For rational functions f, continuous extensions are usually found by __________________________.

Ex. 3 .... (`example10, p79`)

Show that $f(x) = \dfrac{x^2 + x - 6}{x^2 - 4}$, $x \neq 2$ has a continuous extension to $x = 2$, and find that that extension.

sol:

4 Intermediate Value Theorem for Continuous Functions

4.1 *Theorem 11: The Intermediate Value Theorem for Continuous Functions*

> If f is continuous function on a closed interval $[a, b]$ and if y_0 is any value between _________ and _________, then _____________ for some c in $[a, b]$.

圖示如下:

4.2 The Intermediate Value Theorem says that any ________________________ crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.

Ex. 4 .... (`example11, p81`)

Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

sol:

✐ Ex. **5** .... (`example12, p81`)

Use the Intermediate value Theorem to prove that the equation $\sqrt{2x+5} = 4 - x^2$ has a solution.

sol:

實習課練習 (EXERCISE 2.5)

13-25. At what points are the functions in the following continuous?

13. $y = \dfrac{1}{x-2} - 3x.$

18. $y = \dfrac{1}{|x|+1} - \dfrac{x^2}{2}.$

24. $y = \dfrac{\sqrt{x^4+1}}{1+\sin^2 x}.$

25. $y = \sqrt{2x+3}.$

37. Define $g(3)$ in a way that extends $g(x) = (x^2-9)/(x-3)$ to be continuous at $x = 3$.

44. For what value of b is

$$g(x) = \begin{cases} \dfrac{x-b}{b+1} & x < 0, \\ x^2 + b & x > 0 \end{cases}$$

continuous at every x.

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2.6 Limits Involving Infinity; Asymptotes of Graphs

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1 Finite Limits as $x \to \pm\infty$

1.1 *Definitions: Limit as x approaches ∞ or $-\infty$*

(a) We say that $f(x)$ has the **limit L as x approaches infinity** and write

____

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x ______________.

(b) We say that $f(x)$ has the **limit L as x approaches minus infinity** and write ______________ if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x ______________.

1.2 *Theorem 12: Limit Laws as $x \to \pm\infty$*

If L, M, and k are real numbers and $\lim_{x\to\pm\infty} f(x) = L$ and $\lim_{x\to\pm\infty} g(x) = M$, then

(a) sum rule: ______________.

(b) difference rule: ______________.

(c) product rule: ______________.

(d) constant multiple rule: ______________.

(e) quotient rule: ______________.

(f) power rule: if r and s are integers with no commom factor and $s \neq 0$, then ______________, provided that $L^{r/s}$ is a real number.

✐ Ex. **1** .... (`example1, p85`)

Show that (a) $\lim_{x\to\infty} \frac{1}{x} = 0$, (b) $\lim_{x\to-\infty} \frac{1}{x} = 0$.

sol:

✐ Ex. **2** .... (`example2, p85`)

Evaluate (a) $\lim_{x\to\infty} (5 + \frac{1}{x})$, (b) $\lim_{x\to-\infty} \frac{\pi\sqrt{3}}{x^2}$.

sol:

✐ Ex. **3** .... (`example3, p86`)

Evaluate (a) $\lim_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2}$, (b) $\lim_{x\to-\infty} \frac{11x + 2}{2x^3 - 1}$.

sol:

2 Horizontal Asymptotes

2.1 *Definition: Horizontal Asymptote*

> A line $y = b$ is a ____________________ of the graph of a function $y = f(x)$ if either ____________ or ____________.

2.2 Example 1: $f(x) = \dfrac{1}{x}$, the ________ is an asymptote of the curve on the right because __________ and on the left because ____________.

2.3 Example 2: $f(x) = \dfrac{5x^2 + 8x - 3}{3x^2 + 2}$ has the line ________ as a horizontal asymptote on both right and the left because ____________ and ____________.

2.4 The Sandwich Theorem also holds for limits as $x \to \pm\infty$.

Ex. 4 .... (`example4, p87`)

Find the horizontal asymptotes of the graph of $f(x) = \dfrac{x^3 - 2}{|x|^3 + 1}$.

sol:

Ex. 5 .... (`example5, p87`)

(a) Find $\lim\limits_{x\to\infty} \sin(\dfrac{1}{x})$, (b) Find $\lim\limits_{x\to\pm\infty} x \sin(\dfrac{1}{x})$.

sol:

✐ Ex. **6** .... (example6, p87)

Using the Sandwich Theorem, find the horizontal asymptote of the curve $y = 2 + \dfrac{\sin x}{x}$.

sol:

✐ Ex. **7** .... (example7, p88)

Find $\lim\limits_{x\to\infty}(x - \sqrt{x^2 + 16})$.

sol:

3 Oblique Asymptotes

3.1 If the ______________ of a rational function is greater than the ______________ the graph has an ________ or ______________.

3.2 We find an equation for the asymptote by dividing numerator by denominator to express f as a __________ plus a ________ that goes to zero as ________.

Ex. 8 .... (example8, p88)

Find the oblique asymptote of the graph of $f(x) = \dfrac{x^2 - 3}{2x - 4}$.

sol:

4 Infinite Limits

4.1 The limit does not exist: $\lim\limits_{x \to 0^-} \dfrac{1}{x} =$_________, $\lim\limits_{x \to 0^+} \dfrac{1}{x} =$________.

4.2 The limit __________ does not exist.

Ex. 9 .... (example9, p89)

Find $\lim\limits_{x \to 1^+} \dfrac{1}{x-1}$ and $\lim\limits_{x \to 1^-} \dfrac{1}{x-1}$.

sol:

Ex. 10 .... (example11, p90)

(a) $\lim\limits_{x \to 2} \dfrac{(x-2)^2}{x^2 - 4} =$

(b) $\lim\limits_{x \to 2} \dfrac{x-2}{x^2 - 4} =$

(c) $\lim\limits_{x \to 2^+} \dfrac{x-3}{x^2 - 4} =$

(d) $\lim_{x\to 2^-} \frac{x-3}{x^2-4} =$

(e) $\lim_{x\to 2} \frac{x-3}{x^2-4} =$

(f) $\lim_{x\to 2} \frac{2-x}{(x-2)^3} =$

5 Precise Definitions of Infinite Limits

5.1 *Definitions: Infinity, Negative Infinity as Limits*

(a) We say that $f(x)$ approaches ________ as x approaches x_0, and write ________________ if for every positive number B there exists a corresponding $\delta > 0$ such that for all x ________________ $\Rightarrow$ ____________

(b) We say that $f(x)$ approaches ________________ as x approaches x_0, and write ________________ if for every negative number $-B$ there exists a corresponding $\delta > 0$ such that for all x ________________ $\Rightarrow$ ____________

6 Vertical Asymptotes

6.1 *Definitions: Vertical Asymptote*

A line ________ is a vertical asymptote of the graph of a function $y = f(x)$ if either ____________________ or ____________________

Ex. 11 .... (`example13, p91`)

Find the horizontal and vertical asymptotes of the curve $y = \frac{x+3}{x+2}$.

sol:

✐ Ex. **12** .... (`example14, p92`)

Find the horizontal and vertical asymptotes of the graph of $f(x) = -\dfrac{8}{x^2-4}$.

sol:

實習課練習 (EXERCISE 2.6)

10. Find the limit $\lim\limits_{\theta \to -\infty} \dfrac{\cos\theta}{3\theta}$.

16. Find the limit (a) as $x \to \infty$ and (b) as $x \to -\infty$: $f(x) = \dfrac{3x+7}{x^2-2}$.

25. Find the limit: $\lim\limits_{x \to -\infty} \left(\dfrac{1-x^3}{x^2+7x}\right)^5$.

28. Find the limit: $\lim\limits_{x \to \infty} \dfrac{2+\sqrt{x}}{2-\sqrt{x}}$.

41. Find the limit: $\lim\limits_{x \to -8^+} \dfrac{2x}{x+8}$.

44. Find the limit: $\lim\limits_{x \to 0} \dfrac{-1}{x^2(x+1)}$.

53. Find the limits: $\lim \dfrac{1}{x^2-4}$ as
(a) $x \to 2^+$, (b) $x \to 2^-$, (c) $x \to -2^+$, and (d) $x \to -2^-$.

55. Find the limits: $\lim(\dfrac{x^2}{2} - \dfrac{1}{x})$ as
(a) $x \to 0^+$, (b) $x \to 0^-$, (c) $x \to \sqrt[3]{2}$, and (d) $x \to -1$.

61. Find the limits: $\lim(\dfrac{1}{x^{2/3}} + \dfrac{2}{(x-1)^{2/3}})$ as
(a) $x \to 0^+$, (b) $x \to 0^-$, (c) $x \to 1^+$, and (d) $x \to 1^-$.

80. Find the limit: $\lim\limits_{x \to \infty}(\sqrt{x+9} - \sqrt{x+4})$.

THOMAS' CALCULUS (12/E)

3.1 Tangents and the Derivative at a Point

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1 Finding a Tangent to the Graph of a Function

1.1 *Definition: Slope, Tangent Line*

The **slope** of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$m =$ ________________ (provided the limit exists)

The **tangent line** to the curve at P is the line through P with this slope.

1.2 *Finding the Tangent to the Curve $y = f(x)$ at (x_0, y_0)*

(a) Calculate ________ and __________.

(b) Calculate the slope ________________.

(c) If the limit exists, find the tangent line as ________________. (point-slope equation)

Ex. 1 .... (example1, p103)

(a) Find the slope of the curve $y = 1/x$ at $x = a \neq 0$. What is the slope at the point $x = -1$.

(b) Where does the slope equal $-1/4$.

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

sol:

2 Rates of Change: Derivative at a Point

2.1 The difference quotient of f at x_0 with increment h: ____________________.

2.2 If the difference quotient has a limit as h approaches ________, that limit is called the _____________ of f at x_0.

2.3 *Definition*

> The derivative of a function f at a point x_0, denoted $f'(x_0)$, is
>
> ______________________________
>
> provided this limit exists.

2.4 *Summary*

> All the following refer to the same thing.
>
> (a) The _________ of $y = f(x)$ at $x = x_0$.
>
> (b) The _____________________ to the curve $y = f(x)$ at $x = x_0$.
>
> (c) The ________________ of $f(x)$ with respect to x at $x = x_0$.
>
> (d) The ______________________________, $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.
>
> (e) The _____________ of f at $x = x_0$.

實習課練習 (EXERCISE 3.1)

Find an equation for the line tangent to the graph at the given point.

8. $y = \dfrac{1}{x^2},\ (-1, 1).$

11. $f(x) = x^2 + 1,\ (2, 5).$

13. $g(x) = \dfrac{x}{x-2},\ (3, 3).$

18. $f(x) = \sqrt{x+1},\ (8, 3).$

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3.2 The Derivative as a Function

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1 Calculating Derivatives from the Definition

1.1 The slope of a curve $y = f(x)$ at the point where $x = x_0$:

____

this limit is called the ____________ of f at x_0.

1.2 *Definition: Derivative Function*

> The ____________ of the function $f(x)$ with respect to the variable x is the function f' whose value at x is
>
> ___________________________
>
> provided the limit exists.

1.3 Alternative Formula for the Derivative

____

1.4 If _____ exist at a particular x, we say that f is ______________ (has a derivative) at x.

1.5 If f' exists at _____________ in the domain of f, we call f ______________.

1.6 The process of calculating a derivative: _________________.

1.7 $f'(x) =$ __________.

1.8 y: the ____________ variable. x: the ______________ variable.

1.9 Notations

$$f'(x) = \_\_\_\_\_ = \frac{\_\_\_\_\_}{\_\_\_\_\_} = \frac{\_\_\_\_\_}{\_\_\_\_\_} = \frac{\_\_\_\_\_}{\_\_\_\_\_} = \_\_\_\_\_ = \_\_\_\_\_$$

1.10 dy/dx: ______________________________.

1.11 The value of a derivative at a specified number $x = a$.

$$f'(a) = \frac{\_\_\_\_\_}{\_\_\_\_\_} = \frac{\_\_\_\_\_}{\_\_\_\_\_} = \frac{\_\_\_\_\_}{\_\_\_\_\_}.$$

✎ Ex. **1** .... (example1, p107)

Differentiate $f(x) = \dfrac{x}{x-1}$ using the definition.

sol:

✎ Ex. **2** .... (example2, p108)

(a) Find the derivative of $y = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

sol:

2 Differentiable on an Interval; One-Sided Derivative

2.1 A function $y = f(x)$ is ______________ on an ______________ (finite or infinite) if it has a derivative at each point of the interval.

2.2 A function $y = f(x)$ is ________________ on a _________________ $[a, b]$ if it is differentiable on the interior (a, b) and if the limits exist at the endpoints.

2.3 Right-hand derivative at a:

____

2.4 Left-hand derivative at b:

____

✐ Ex. **3** .... (`example4, p110`)

Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

sol:

✐ Ex. **4** .... (`example5, p110`)

Show that the function $y = \sqrt{x}$ is not differentiable at $x = 0$.

sol:

3 Differentiable Functions are Continuous

3.1 A function has a derivative at a point x_0 if the ______________________ through $P(x_0, f(x_0))$ and a nearby point Q on the graph approach a ________ as Q approaches P.

3.2 When does a function not have a derivative at a point?

(a) a _________, where the one-sided derivatives differ.

(b) a ________, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.

(c) a ________________, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).

(d) a ________________.

3.3 A function is continuous at every point where it has a ____________.

3.4 *Theorem 1: Differentiability Implies Continuity*

If f has a ____________ at $x = c$, then f is ____________ at $x = c$.

Proof:

3.5 A function **need not** have a derivative at a point where it is continuous.

實習課練習 (EXERCISE 3.2)

Using the definition, calculate the derivative of the function in Exercise 2 and 4. Then find the values of the derivatives as specified.

2. $F(x) = (x-1)^2 + 1; F'(-1), F'(0), F'(2)$.

4. $k(z) = \dfrac{1-z}{2z}; k'(-1), k'(1), k'(\sqrt{2})$.

13. Differentiate the functions and find the slope of the tangent line at the given value of the independent variable. $f(x) = x + \dfrac{9}{x}, x = -3$.

17. Differentiate the function and find the equation of the tangent line at the indicated point on the graph of the function. $y = f(x) = \dfrac{8}{\sqrt{x-2}},\ (x, y) = (6, 4)$.

26. Use the alternative formula for derivative to find the derivative of the function: $g(x) = 1 + \sqrt{x}$.

41. determine if the piecewise defined function is differentiable at the origin.

$$f(x) = \begin{cases} 2x - 1, & x \geq 0 \\ x^2 + 2x + 7, & x < 0 \end{cases}$$

61. Derivative of $y = |x|$. Graph the derivative of $f(x) = |x|$. Then graph $y = (|x| - 0)/(x - 0) = |x|/x$. What can you conclude?

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3.3 Differentiation Rules

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1 Powers, Multiples, Sums, and Differences

1.1 *RULE 1: Derivative of a Constant Function*

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \underline{\hspace{3cm}} = \underline{\hspace{1.5cm}}$$

Proof:

1.2 *RULE 2: Power Rule for Positive Integers*

If n is a positive integer, then

$$\frac{d}{dx}x^n = \underline{\hspace{3cm}}$$

Proof:

1.3 *Power Rule (General Version)*

If n is any real number, then

$$\frac{d}{dx}x^n = \underline{\hspace{3cm}}$$

for all x where the power x^n and x^{n-1} are defined.

1.4 *RULE 3: Constant Multiple Rule*

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = \underline{\hspace{3em}}$$

1.5 *RULE 4: Derivative Sum Rule*

If u and v are differentiable functions of x, then their sum $u+v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u+v) = \underline{\hspace{4em}}$$

✐ Ex. **1** .... (`example1, p117`)

Differentiate the following powers of x. (a) x^3, (b) $x^{2/3}$, (c) $x^{\sqrt{2}}$, (d) $\frac{1}{x^4}$, (e) $x^{-4/3}$, (f) $\sqrt{x^{2+\pi}}$.

sol:

✐ Ex. **2** .... (`example3, p118`)

Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

sol:

2 Products

2.1 *RULE 5: Derivative Product Rule*

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = \_\_\_\_\_\_\_\_\_\_$$

2.2 $\frac{d}{dx}[f(x)g(x)] = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

Proof:

✎ Ex. 3 .... (example5, p119)

Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

sol:

3 Quotients

3.1 *RULE 6: Derivative Quotient Rule*

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}(\frac{u}{v}) = \_\_\_\_\_\_\_\_\_\_$$

3.2 $\frac{d}{dx}[\frac{f(x)}{g(x)}] = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

Proof:

Ex. 4 .... (example6, p120)

Find the derivative of $y = \dfrac{t^2-1}{t^2+1}$.

sol:

4 Second- and Higher-Order Derivatives

4.1 The function _______ is called the ___________________ of f because it is the derivative of the first derivative.

$$f''(x) = \rule{8cm}{0.4pt}$$

4.2 $f^{(n)}(x)$: the ________________ of y with respect to x for any positive integer n.

Ex. 5 .... (example8, p122)

Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

sol:

實習課練習 (EXERCISE 3.3)

5. Find the first and second derivatives of $y = \frac{4x^3}{3} - x$.

8. Find the first and second derivatives of $s = -2t^{-1} + \frac{4}{t^2}$.

15. Find y' of $y = (x^2+1)(x+5+\frac{1}{x})$.

21. Find the derivative of $v = (1-t)(1+t^2)^{-1}$.

24. Find the derivative of $u = \frac{5x+1}{2\sqrt{x}}$.

28. Find the derivative of $y = \frac{(x+1)(x+2)}{(x-1)(x-2)}$.

34. Find the first and second derivatives of $s = \frac{t^2+5t-1}{t^2}$.

38. Find the first and second derivatives of $w = (z+1)(z-1)(z^2+1)$.

53(a). Find an equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1, 0)$

55. Evaluate $\lim_{x\to 1} \frac{x^{50}-1}{x-1}$.

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3.5 Derivatives of Trigonometric Functions

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1 Derivative of the Sine, Cosine Function

1.1 *The derivative of the sine function is the cosine function*

$$\frac{d}{dx}(\sin x) = \underline{\qquad\qquad}$$

Proof:

1.2 *The derivative of the cosine function is the sine function*

$$\frac{d}{dx}(\cos x) = \underline{\qquad\qquad}$$

Proof:

Ex. **1** .... (example1, p136)

Find $\dfrac{dy}{dx}$ if (a) $y = x^2 - \sin x$, (b) $y = x^2 \sin x$, (c) $y = \dfrac{\sin x}{x}$.

sol:

Ex. **2** .... (example2, p137)

Find $\dfrac{dy}{dx}$ if (a) $y = 5x + \cos x$, (b) $y = \sin x \cos x$, (c) $y = \dfrac{\cos x}{1 - \sin x}$.

sol:

2 Derivatives of the Other Basic Tri. Functions

2.1 *Derivatives of the Other Trigonometric Functions*

(a) $\dfrac{d}{dx}(\tan x) =$ ________. (b) $\dfrac{d}{dx}(\cot x) =$ ________.

(c) $\dfrac{d}{dx}(\sec x) =$ ________. (d) $\dfrac{d}{dx}(\csc x) =$ ________.

✎ Ex. **3** .... (example5, p139)

Find $\frac{d}{dx}(\tan x)$

sol:

✎ Ex. **4** .... (example6, p139)

Find y'' if $y = \sec x$.

sol:

實習課練習 (EXERCISE 3.5)

2. Find $\frac{dy}{dx}$ if $y = \frac{3}{x} + \sin x$.

7. Find $\frac{dy}{dx}$ if $y = f(x) = \sin x \tan x$.

9. Find $\frac{dy}{dx}$ if $y = (\sec x + \tan x)(\sec x - \tan x)$.

12. Find $\frac{dy}{dx}$ if $y = \frac{\cos x}{1 + \sin x}$.

21. Find $\frac{ds}{dt}$ if $s = \frac{1 + \csc t}{1 - \csc t}$.

26. Find $\frac{dr}{d\theta}$ if $r = (1 + \sec\theta)\sin\theta$.

29. Find $\frac{dp}{dq}$ if $p = \frac{\sin q + \cos q}{\cos q}$.

34. Find $y^{(4)} = \frac{d^4y}{dx^4}$ if (a) $y = -2\sin x$, (b) $y = 9\cos x$.

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3.6 The Chain Rule

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1 Derivative of a Composite Function

1.1 *Theorem 3: The Chain Rule*

> If $f(u)$ is differentiable at the point $u =$ ________ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and
> $$(f \circ g)'(x) = \underline{\hspace{4cm}}.$$

1.2 If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} =$ __________, where dy/du is evaluated at $u = g(x)$.

1.3 Think of the derivative as a rate of change: if $y = f(u)$ changes half as fast as u and $u = g(x)$ changes three times as fast as x, then we expect y to change ________ times as fast as x.

1.4 *Repeated Use of the Chain Rule*

> $$\frac{dy}{dx} = \underline{\hspace{3cm}}$$

Ex. 1 .... (`example1, p142`)

The function $y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$ is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculate $\frac{dy}{du}$, $\frac{du}{dx}$, and $\frac{dy}{du} \cdot \frac{du}{dx}$. Use the Chain Rule to calculate $\frac{dy}{dx}$.

sol:

✎ Ex. **2** .... (example3, p144)

Differentiate $\sin(x^2 + x)$ with respect to x.

sol:

✎ Ex. **3** .... (example4, p145)

Find the derivative of $g(t) = \tan(5 - \sin 2t)$

sol:

2 The Power Chain Rule

2.1 *The Chain Rule with Powers of a Function*

If n is a positive or negative integer, $f(u) = u^n$, and u is a differentiable function of x, then

$$\frac{d}{dx}u^n = \underline{\qquad\qquad}$$

✎ Ex. **4** .... (`example5, p145`)

(a) $\dfrac{d}{dx}(5x^3 - x^4)^7$, (b) $\dfrac{d}{dx}(\dfrac{1}{3x-2})$, (c) $\dfrac{d}{dx}(\sin^5 x)$

sol:

✎ Ex. **5** .... (`example6, p146`)

$\dfrac{d}{dx}(|x|)$.

sol:

實習課練習 (EXERCISE 3.6)

1. Find $\dfrac{dy}{dx}$ if $y = 6u - 9$, $u = (1/2)x^4$.

6. Find $\dfrac{dy}{dx}$ if $y = \sin u$, $u = x - \cos x$.

9. Find $\dfrac{dy}{dx}$ if $y = (2x+1)^5$.

16. Find $\dfrac{dy}{dx}$ if $y = \cot(\pi - \dfrac{1}{x})$.

20. Find the derivative of $q = \sqrt{2r - r^2}$.

35. Find the derivative of $f(\theta) = (\dfrac{\sin\theta}{1+\cos\theta})^2$.

41. Find dy/dt of $y = \sin^2(\pi t - 2)$.

53. Find dy/dt of $y = \sqrt{1 + \cos(t^2)}$.

60. Find y'' of $y = (1 - \sqrt{x})^{-1}$.

67. Find the value of $(f \circ g)'$ at the given value of x.
$f(u) = \cot \dfrac{\pi u}{10}$, $u = g(x) = 5\sqrt{x}, x = 1$.

70. Find the value of $(f \circ g)'$ at the given value of x.
$f(u) = (\dfrac{u-1}{u+1})^2$, $u = g(x) = \dfrac{1}{x^2} - 1, x = -1$.

81(a). Find the tangent to the curve $y = 2\tan(\pi x/4)$ at $x = 1$.

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3.7 Implicit Differentiation

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1 Implicit Differentiation

1.1 An equation of the form __________: express y explicitly in terms of the variable x.

1.2 The parametric equation: __________ and __________.

1.3 An _______________ between the variables x and y:
e.g., $x^2 + y^2 = 25$, $y^2 - x = 0$, or $x^3 + y^3 - 9xy = 0$.

1.4 *Implicit Differentiation*

(a) Differentiate ___________ of the equation with respect to x, treating y as a differentiable function of x.

(b) Collect the term with ________ on one side of the equation.

(c) Solve for ________.

Ex. 1 . . . . (example1, p150)

Find dy/dx if $y^2 = x$.

sol:

✐ Ex. **2** .... (example2, p150)

Find the slope of circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

sol:

✐ Ex. **3** .... (example3, p151)

Find dy/dx if $y^2 = x^2 + \sin xy$.

sol:

✐ Ex. **4** .... (example4, p152)

Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

sol:

實習課練習 (EXERCISE 3.7)

1. Use implicit differentiation to find dy/dx if $x^2y + xy^2 = 6$.

5. Use implicit differentiation to find dy/dx if $x^2(x-y)^2 = x^2 - y^2$.

8. Use implicit differentiation to find dy/dx if $x^3 = \dfrac{2x-y}{x+3y}$.

13. Use implicit differentiation to find dy/dx if $y\sin(\dfrac{1}{y}) = 1 - xy$.

24. Use implicit differentiation to find dy/dx and d^2y/dx^2 if $xy + y^2 = 1$.

38(a). Find the line that is tangent to the curve at the given points. $x^2\cos^2 y - \sin y = 0$, $(0, \pi)$.

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3.9 Linearization and Differentials

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1 Linearization

1.1 *Definitions: Linearization, Standard Linear Approximation*

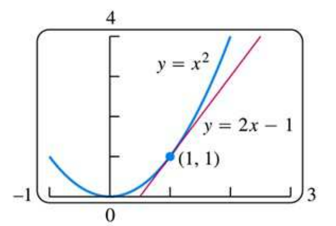
(a) If f is differentiable at $x = a$, then the approximating function

$$L(x) = \underline{\hspace{5cm}}$$

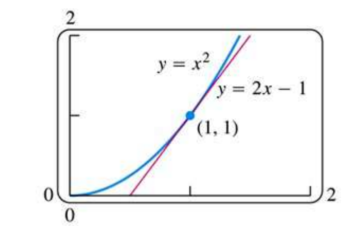
is the ________________ of f at a.

(b) The approximation $f(x) \approx L(x)$ of f by L is the __________________ __________________ of f at a.

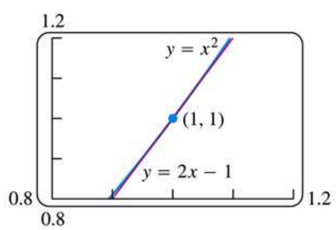
(c) The point $x = a$ is the __________ of the approximation.



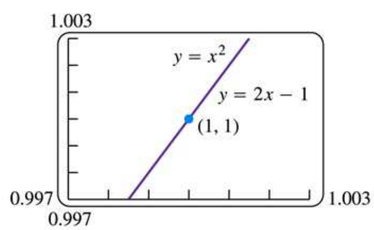
$y = x^2$ and its tangent $y = 2x - 1$ at (1, 1).



Tangent and curve very close near (1, 1).



Tangent and curve very close throughout entire x-interval shown.



Tangent and curve closer still. Computer screen cannot distinguish tangent from curve on this x-interval.

✎ Ex. **1** .... (`example1, p165`)

Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$.

sol:

✎ Ex. **2** .... (`example2, p166`)

Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 3$.

sol:

✎ Ex. **3** .... (`example3, p166`)

Find the linearization of $f(x) = \cos x$ at $x = \pi/2$.

sol:

2 Differentials

1. *Definition: Differential*

   If $y = f(x)$ be a differentiable function. The ______________ dx is an independent variable. The differential dy is

   $$dy = \underline{\hspace{3cm}}$$

2. dy is always a ______________ variable. It depends on both x and dx.

Ex. 4 .... (`example4, p167`)

(a) Find dy if $y = x^5 + 37x$, (b) find the value of dy when $x = 1$ and $dx = 0.2$.

sol:

Ex. 5 .... (`example5, p168`)

(a) $d(\tan 2x)$, (b) $d(\dfrac{x}{x+1})$.

sol:

實習課練習 (EXERCISE 3.9)

1. Find the linearization $L(x)$ of $f(x) = x^3 - 2x + 3$ at $x = 2$.

4. Find the linearization $L(x)$ of $f(x) = \sqrt[3]{x}$ at $x = -8$.

6(a). Find the linearization $L(x)$ of $f(x) = \sin x$ at $x = 0$.

17. Find dy if $y = x^3 - 3\sqrt{x}$.

20. Find dy if $y = \frac{2\sqrt{x}}{3(1+\sqrt{x})}$.

21. Find dy if $2y^{3/2} + xy - x = 0$.

28. Find dy if $y = 2\cot(\dfrac{1}{\sqrt{x}})$.

30. $f(x) = 2x^2 + 4x - 3, x_0 = -1, dx = 0.1$. Each function $f(x)$ changes values when x changes from x_0 to $x_0 + dx$. Find

(a) the change $\Delta f = f(x_0 + dx) - f(x_0)$;

(b) the value of the estimate $df = f'(x_0)\, dx$; and

(c) the approximation error $|\Delta f - df|$.

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4.1 Extreme Values of Functions

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1 Global (Absolute) Extreme Values

1.1 *Definitions: Absolute Maximum, Absolute Minimum*

Let f be a function with domain D.Then

(a) f has an ________________ on D at a point c if __________ for all x in D, and

(b) an ________________ on D at c if __________ for all x in D.

1.2 Absolute maximum and mimimum values are called ____________ or ____________.

1.3 *Theorem 1: The Extreme Value Theorem*

If f is __________ on a ____________ $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) =$ ______, $f(x_2) =$ ______, and ____________ for every other x in $[a, b]$.

1.4 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$. (圖示如下)

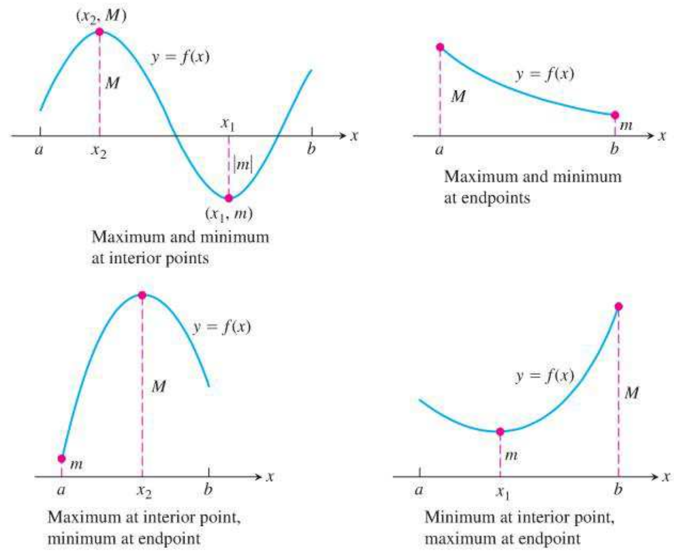

FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

✎ Ex. 1 .... (example1, p185)

Functions with the same defining rule can have different extrema, depending on the domain.

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	________________.
		________________________.
(b) $y = x^2$	$[0, 2]$	________________________.
		________________________.
(c) $y = x^2$	$(0, 2]$	________________________.
		________________.
(d) $y = x^2$	$(0, 2)$	________________.

2 Local (Relative) Extreme Values

2.1 *Definitions: Local Maximum, Local Minimum*

(a) A function f has a ________________ at an ____________ c of its domain if ____________ for all x in some open interval containing c.

(b) A function f has a ________________ at an ____________ c of its domain if ____________ for all x in some open interval containing c.

2.2 Local extrema are also called ________________.

2.3 An absolute maximum is also a ________________.

2.4 How to classify maxima and minima: (圖示如下)

3 Finding Extrema

3.1 *Theorem 2: The First Derivative Theorem for Local Extreme Values*

> If f has a ________________ or ________________ at an interior point c of its domain, and if f' is defined at c, then ____________.

3.2 A function's ________________ is always ________ at an interior point where the function has a ____________________ and the derivative is defined.

3.3 Where a function f can possibly have an extreme value (local or global) are

(a) the interior points where __________,

(b) the interior points where ________________.

(c) the ____________ of the domain of f.

3.4 *Definition: Critical Point*

> An _______________ of the domain of a function f where f' is ________ or ____________ is a _______________ of f.

3.5 The only domain points where a function can assume extreme values are ________________ and ____________.

3.6 How to find the **absolute extrema** of a continuous function f on a finite closed interval.

(a) Evaluate f at all ________________ and ____________.

(b) Take the __________ and ___________ of these values.

Ex. 2 .... (example2, p188)

Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

sol:

Ex. 3 .... (example3, p189)

Find the absolute extrema values of $g(t) = 8t - t^4$ on $[-2, 1]$.

sol:

Ex. 4 .... (example4, p189)

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on $[-2, 3]$.

sol:

實習課練習 (EXERCISE 4.1)

29. Find the absolute maximum and minimum values of $g(x) = \sqrt{4 - x^2}$ on $-2 \le x \le 1$.

31. Find the absolute maximum and minimum values of $f(\theta) = \sin\theta$ on $-\frac{\pi}{2} \le \theta \le \frac{5\pi}{6}$.

36. Find the absolute maximum and minimum values of $f(t) = |t - 5|$ on $4 \le t \le 7$.

58. Find the extreme values of $y = \dfrac{x+1}{x^2 + 2x + 2}$ and where they occur.

64. Find the derivative at each critical point and determine the local extreme values.
$$y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \ge 0 \end{cases}$$

68. Determine all extrema of $f(x) = |x^3 - 9x|$.

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4.2 The Mean Value Theorem

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1 Rolle's Theorem

1.1 *Theorem 3: Rolle's Theorem*

> Suppose that $y = f(x)$ is ______________ at every point of the closed interval $[a, b]$ and ________________ at every point of its interior (a, b). If _______________, then there is at least one number c in (a, b) at which ____________.

1.2 Rolle's Theorem says that a differentiable curve had at least one ____________________ between any two points where it crosses a _________________. It may have just one, or more. (圖示如下)

2 The Mean Value Theorem

2.1 *Theorem 4: The Mean Value Theorem*

> Suppose that $y = f(x)$ is ______________ on a closed interval $[a, b]$ and ________________ on the interval's interior (a, b). Then there is at least one point c in (a, b) at which ______________________.

2.2 Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one ___________ parallel to chord AB. (圖示如下)

2.3 *A Physical Interpretation*

If we think of the number ______ as the ______ in f over $[a, b]$ and ______ as an ______, then the Mean Value Theorem says that at ______ the instantaneous change must ______ the average change over the entire interval.

2.4 *Mathematical Consequences*

(a) If ______ at each point x of an open interval (a, b), then ______ for all $x \in (a, b)$, where C is a constant.

(b) If ______ at each point x in an open interval (a, b), then there exists a constant C such that ______ for all $x \in (a, b)$. That is, ______ is a constant on (a, b).

Ex. 1 . . . . (`example1, p193`)

Show that the equation $x^3 + 3x + 1 = 0$ has exactly one real solution.

sol:

Ex. 2 . . . . (`example2, p194`)

Find the values of c that satisfy the Mean Value Theorem for $f(x) = x^2$ on $[0, 2]$.

sol:

實習課練習 (EXERCISE 4.2)

Find the values of c that satisfy the Mean Value Theorem for the functions and intervals.

1. $f(x) = x^2 + 2x - 1$, $[0, 1]$.

3. $f(x) = x + \dfrac{1}{x}$, $[1/2, 2]$.

4. $f(x) = \sqrt{x-1}$, $[1, 3]$.

14. For what values of a, m and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypothesis of the Mean value Theorem on the interval $[0, 2]$?

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4.3 Monotonic Functions and The First Derivative Test

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1 Increasing Functions and Decreasing Functions

1.1 What kinds of functions have positive derivatives or negative derivatives?

(a) The only functions with positive derivatives are __________ functions.

(b) The only functions with negative derivatives are __________ functions.

1.2 *Definitions: Increasing, Decreasing Function*

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

(a) If __________ whenever then __________ is said to be __________ on I.

(b) If __________ whenever then __________ is said to be __________ on I.

A function that is increasing or decreasing on I is called __________ on I.

1.3 **Example:**

(a) The function $f(x) = x^2$ decreases on __________ and increases on __________.

(b) The function $f(x) = x^2$ is monotonic on __________ and __________ but it is not monotonic on __________. (圖示如下)

1.4 *COROLLARY 3: First Derivative Test for Monotonic Functions*

Suppose that f is ____________ on $[a, b]$ and ______________ on (a, b).

(a) If ___________ at each point $x \in (a, b)$, then f is ____________ on $[a, b]$.

(b) If ___________ at each point $x \in (a, b)$, then f is ____________ on $[a, b]$.

Ex. 1 .... (`example1, p199`)

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing.

sol:

2 First Derivative Test for Local Extrema

2.1 *First Derivative Test for Local Extrema*

Suppose that c is a ______________ of a continuous function f and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

(a) If f' changes ______________ at c, then f has a ________________ at c;

(b) If f' changes from __________ at c, then f has a ________________ at c;

(c) If does not change sign at c (that is, is positive on both sides of c or negative on both sides), then f has ___________________ at c.

2.2 A function's first derivative tells how the graph rises and falls.

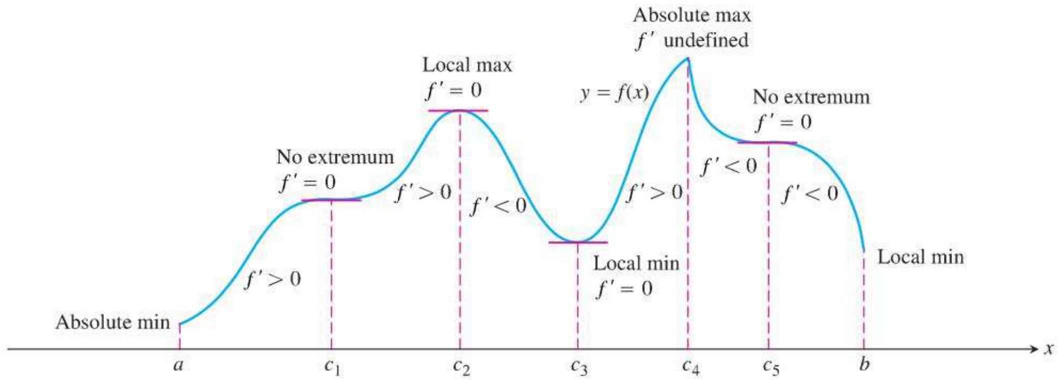

FIGURE 4.21 The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.

✎ Ex. **2** .... (`example2, p201`)

Find the critical points of $f(x) = x^{1/3}(x-4)$. Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

sol:

實習課練習 (EXERCISE 4.3)

Answer the following questions about the functions whose derivatives are given in Exercise 1-14.

(a) What are the critical points of f?

(b) On what intervals is f increasing or decreasing?

(c) At what points, if any, does f assume local maximum and minimum values?

4. $f'(x) = (x-1)^2(x+2)^2$.

11. $f'(x) = x^{-1/3}(x+2)$.

In Exercise 15-40.

(a) Find the open intervals on which the function is increasing and decreasing.

(b) Identify the function's local extreme values, if any, saying where they occur.

23. $f(\theta) = 3\theta^2 - 4\theta^3$.

34. $g(x) = x^2\sqrt{5-x}$.

36. $f(x) = \dfrac{x^3}{3x^2+1}$.

39. $h(x) = x^{1/3}(x^2-4)$.

69. Determine the values of constants a and b so that $f(x) = ax^2 + bx$ has an absolute maximum at the point $(1, 2)$.

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4.4 Concavity and Curve Sketching

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1 Concavity

1.1 *Definitions: Concave Up, Concave Down*

The graph of a differentiable function $y = f(x)$

(a) is ____________ on an open interval I if ______ is ____________ on I

(b) is ____________ on an open interval I if ______ is ____________ on I.

1.2 *The Second Derivative Test for Concavity*

Let $y = f(x)$ be twice-differentiable on an interval I.

(a) If __________ on I, the graph of f over I is ____________.

(b) If __________ on I, the graph of f over I is ____________.

1.3 **Example:**

(a) The graph of $f(x) = x^3$ is concave down on __________ and concave up on __________.

(b) The curve $y = x^2$ is concave up on ____________ because its second derivative $y'' = 2$ is always positive. (圖示如下)

Ex. 1 .... (example2, p204)

Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

sol:

2 Points of Inflection

2.1 *Definitions: Point of Inflection*

A point where the graph of a function has a ____________ and where the ________________ is a point of __________.

2.2 **Example:**

(a) The curve $y = 3 + \sin x$ changes concavity at the point ________. We call ________ point of inflection of the curve.

(b) The curve $y = x^4$ has ________________ at $x = 0$. Even though $y'' = 12x^2$ is zero there, it does not change sign.

(c) The curve $y = x^{1/3}$ has a point of inflection at $x = 0$, but does not exist there.

$$y'' = \underline{\qquad\qquad} = \underline{\qquad\qquad} = \underline{\qquad\qquad}$$

3 Second Derivative Test for Local Extrema

3.1 *Theorem: Second Derivative Test for Local Extrema*

Suppose f is continuous on an open interval that contains

(a) If __________ and __________ then f has a ______________ at $x = c$.

(b) If __________ and __________ then f has a ______________ at $x = c$.

(c) If __________ and __________ then the test fails. The function f may have a local maximum, a local minimum, or neither.

3.2 *Strategy for Graphing $y = f(x)$*

(a) Identify the ____________ of f and any ______________ the curve may have.

(b) Find ______ and ______.

(c) Find the ________________ of f and identify the function's behavior at each one.

(d) Find where the curve is______________ and where it is ______________.

(e) Find the ____________________, if any occur, and determine the ____________ of the curve.

(f) Identify any ______________.

(g) Plot ______________, such as the intercepts and the points found in Steps (c)-(e), and sketch the curve.

Ex. 2 .... (`example7, p207`)

Sketch a graph of the function $f(x) = x^4 - 4x^3 = 10$ using the following steps.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

sol:

✎ Ex. **3** .... (example8, p208)

Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

sol:

✎ Ex. **4** .... (example9, p209)

Sketch the graph of $f(x) = \dfrac{x^2+4}{2x}$.

sol:

實習課練習 (EXERCISE 4.4)

Graph the equations. Include the coordinates of any local extreme points and inflection points.

21. $y = x^4(x-5)$.

23. $y = x + \sin x$, $0 \le x \le 2\pi$.

46. $y = |x^2 - 2x|$.

83. Graph the rational function: $y = \dfrac{x^2}{x+1}$.

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4.5 Applied Optimization

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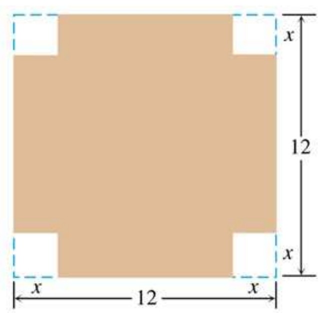
1 Solving Applied Optimization Problems

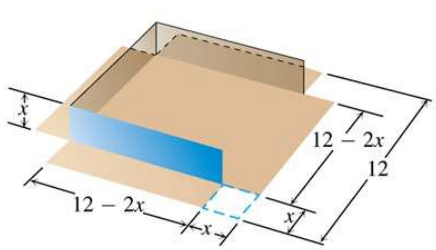
Ex. **1** .... (example1, p214)

An open-top box is to be made by cutting small congruent squares from the corners of a 12cm by 12cm sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible.

sol:



(a)

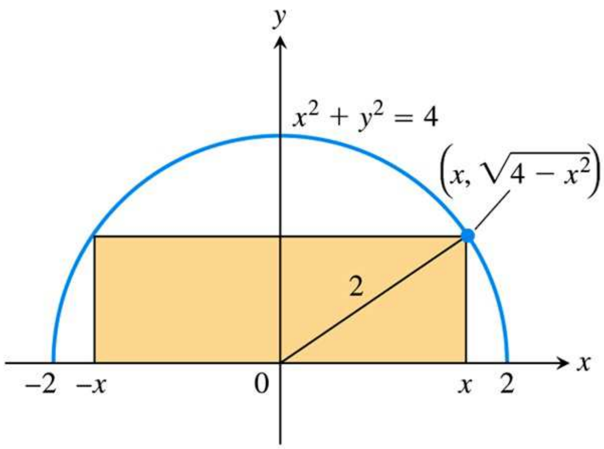


(b)

✐ Ex. **2** .... (`example3, p216`)

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

sol:



實習課練習 (EXERCISE 4.5)

4. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

13. Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area?

30. Find a positive number for which the sum of it and its reciprocal and four times its square is the smallest possible.

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4.6 Newton's Method

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1 Procedure for Newton's Method

1.1 Newton's Method (also called ____________ Method) is a technique to ____________ the solution to an equation ________.

1.2 Essentially it uses ____________ in place of the graph of $y = f(x)$ near the points where ________.

1.3 A value of x where ________ is a ______ of the function f and a ________ of the equation $f(x) = 0$.

1.4 *Procedure for Newton's Method*

(a) Guess a ______________ to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.

(b) Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = \underline{\qquad\qquad\qquad}, \quad \text{if } f'(x_n) \neq 0.$$

圖示說明如下:

Ex. 1 .... (example1, p226)

Find the positive root of the equation $f(x) = x^2 - 2 = 0$.
(We find decimal approximations to $\sqrt{2}$ by estimating the positive root of the equation $f(x) = x^2 - 2 = 0$.)

sol:

Ex. 2 .... (example2, p227)

Find the x-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

sol:

實習課練習 (EXERCISE 4.6)

2. Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2.

5. Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find x_2.

6. Use Newton's method to find the negative fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = -1$ and find x_2.

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4.7 Antiderivatives

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1 Finding Antiderivatives

1.1 We want to find a function F from its ______________. ________________ If such a function F exists, it is called an ________________ of f.

1.2 *Definition: Antiderivative*

> A function F is an antiderivative of f on an interval I if ________________ for all x in I.

1.3 If F is an ________________ of f on an interval I, then the most general antiderivative of f on I is ____________ where C is an arbitrary constant.

1.4 *Definition: Indefinite Integral, Integrand*

> The set of all antiderivatives of f is the ____________________ of f with respect to x, denoted by_____________. The symbo l ______ is an integral sign. The function f is the _____________ of the ___________ and x is the variable of ______________.

Ex. 1 .... (`example1, p230`)

Find an antiderivative for each of the following functions.

1. $f(x) = 2x$.
2. $g(x) = \cos x$.
3. $h(x) = 2x + \cos x$.

sol:

NOTE: ____________________, __________________________,
______________________________.

Ex. 2 . . . . (example2, p231)

Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

sol:

Ex. 3 . . . . (exampleTABLE4.2, p232)

Function	General antiderivative
1. x^n	____________, $n \neq -1$, n rational.
2. $\sin kx$	____________, k a constant, $k \neq 0$.
3. $\cos kx$	____________, k a constant, $k \neq 0$.
4. $\sec^2 x$	____________.
5. $\csc^2 x$	____________.
6. $\sec x \tan x$	____________.
7. $\csc x \cot x$	____________.

Ex. 4 . . . . (example3, p231)

Find an general antiderivative of each of the following functions.

(a) $f(x) = x^5$.

(b) $g(x) = \dfrac{1}{\sqrt{x}}$.

(c) $h(x) = \sin 2x$.

(d) $i(x) = \cos \dfrac{x}{2}$.

sol:

2 Indefinite Integrals

2.1 *Definition*

The collection of all ________________ of f is called the __________________ of f with respect to x, and is denoted by

____

The symbol $\int$ is an _______________. The function f is the _____________ of the integral, and x is the variable of integration.

Ex. 5 . . . . (example6, p235)

Evaluate $\int (x^2 - 2x + 5)\, dx$.

sol:

實習課練習 (EXERCISE 4.7)

Finding antiderivatives.

6. (a) $-\dfrac{2}{x^3}$ (b) $\dfrac{1}{2x^3}$, (c) $x^3 - \dfrac{1}{x^3}$.

9. (a) $\dfrac{2}{3}x^{-1/3}$ (b) $\dfrac{1}{3}x^{-2/3}$, (c) $-\dfrac{1}{3}x^{-4/3}$.

15. (a) $\csc x \cot x$ (b) $csc5x \cot 5x$ (c) $-\pi \csc \dfrac{\pi x}{2} \cot \dfrac{\pi x}{2}$

Finding indefinite integrals.

28. $\displaystyle\int (\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}})\ dx$

32. $\displaystyle\int x^{-3}(x+1)\ dx$

37. $\displaystyle\int 7\sin\frac{\theta}{3}\ d\theta$

42. $\displaystyle\int \frac{2}{5}\sec\theta\tan\theta\ d\theta$

50. $\displaystyle\int (2+\tan^2\theta)\ d\theta$

53. $\displaystyle\int \cos\theta(\tan\theta+\sec\theta)\ d\theta$

THOMAS' CALCULUS (12/E)

5.1∼ 5.2 Estimating with Finite Sums, Limits of Finite Sums

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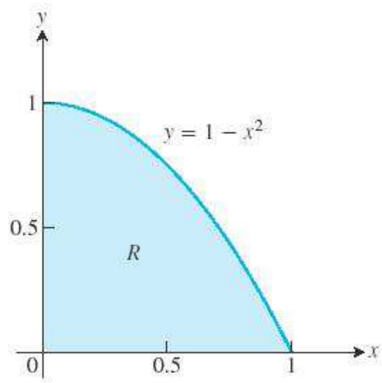
1 Approximating Area

1.1 What is the area of the shaded region R that lies above the x-axis, below the graph of $y = 1 - x^2$ and between the vertical lines $x = 0$ and $x = 1$? (圖示如下:)



(a) **upper sum**:

(b) **lower sum**:

(c) **midpoint rule**:

2 Finite Sums and Sigma Notation

2.1 $a_1 + a_2 + \cdots + a_{n-1} + a_n =$ ________ $=$ ________ .

2.2 Examples

(a) $\displaystyle\sum_{k=1}^{5} k =$ ____________________

(b) $\displaystyle\sum_{k=1}^{3} (-1)^k k =$ ______________________________

(c) $\displaystyle\sum_{k=1}^{2} \frac{k}{k+1} =$ __________________

(d) $\displaystyle\sum_{i=4}^{5} \frac{i^2}{i-1} =$ ___________________

2.3 Algebra Rules for Finite Sums

(a) Sum Rule: $\sum(a_k + b_k) =$ _____________

(b) Difference Rule: $\sum(a_k - b_k) =$ _____________

(c) Constant Multiple Rule: $\sum ca_k =$ _________ (Any number c)

(d) Constant Value Rule: $\sum c =$ _______

2.4 Examples:

(a) $\sum_{k=1}^{n}(3k - k^2) =$

(b) $\sum_{k=1}^{n}(-a_k) =$

(c) $\sum_{k=1}^{3}(k + 4) =$

(d) $\displaystyle\sum_{k=1}^{n} \frac{1}{n} =$

Ex. 1 .... (`example4, p258`)

Show that $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

sol:

NOTE:

1. The first n squares: $\sum_{k=1}^{n} k^2 =$ ______________________.

2. The first n cubes: $\sum_{k=1}^{n} k^3 =$ __________________.

3 Riemann Sums

3.1 Let $n-1$ points $\{x_1, x_2, \cdots, x_{n-1}\}$ between a and b and satisfying

____

3.2 A **partition** of $[a, b]$: ________________________________.

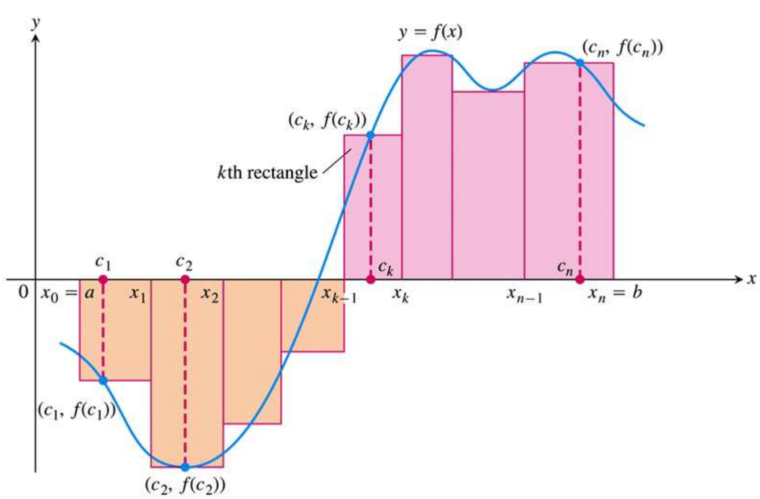
3.3 The kth subinterval of P is _______________.

3.4 The ___________ of a partition P, ___________, the largest of all subinterval widths.

3.5 A ____________________ for f on the interval $[a, b]$:

____

for every $c_k \in [x_{k-1}, x_k]$, $k = 1, \cdots, n$. (圖示如下)



實習課練習 (EXERCISE 5.1)

3. Use finite approximation to estimate the area under the graph of the function $f(x) = \dfrac{1}{x}$ using (a) a lower sum; (b) an upper sum; and (c) midpoint rule with four rectangles of equal width.

實習課練習 (EXERCISE 5.2)

21. Evaluate $\displaystyle\sum_{k=1}^{7}(-2k)$.

26. Evaluate $\displaystyle\sum_{k=1}^{7} k(2k+1)$.

27. Evaluate $\displaystyle\sum_{k=1}^{5} \frac{k^3}{225} + (\sum_{k=1}^{5} k)^3$.

32. Evaluate (a) $\displaystyle\sum_{k=1}^{5} \frac{k^3}{225} + (\sum_{k=1}^{5} k)^3$, (b) $\displaystyle\sum_{k=1}^{n} \frac{c}{n}$, (c) $\sum_{k=1}^{n} \frac{k}{n^2}$.

34. Graph $f(x) = -x^2$ on $[0, 1]$. Partition the interval into four subintervals of equal length. Add to your sketch the rectangles associated with the Riemann sum $\sum_{k=1}^{4} f(c_k)\Delta x_k$, given that c_k is the midpoint of the kth subinterval.

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5.3 The Definite Integral

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1 Limits of Riemann Sums

1.1 *Definitions: The Definite Integral as a Limit of Riemann Sums*

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number I is the __________ of f over $[a, b]$ and that I is the ________ of the __________ if the following condition is satisfied:

Given any number $\delta > 0$ there is a corresponding number such that for every partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ with ________ and any choice of c_k in $[x_k, x_{k+1}]$, we have

____

1.2 The symbol for the number I in the definition of the definite integral is

____

which is read as "____________________" or sometimes as "____________________"

1.3 When the definition is satisfied, we say the Riemann sums of f on $[a, b]$ ________ to the definite integral $I = \int_a^b f(x)\, dx$ and that f is ________ over $[a, b]$.

____

1.4 When each partition has n equal subintervals, each of width ________________,

____

1.5 The __________________: t, u, x: $\int_a^b f(t)\ dt$, $\int_a^b f(u)\ du$, $\int_a^b f(x)\ dx$.

1.6 *Theorem 1: The Existence of Definite Integrals*

> A ____________ function is integrable. That is, if a function f is ____________ on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

2 Properties of Definite Integrals

2.1 When f and g are integrable, the definite integral satisfies following.

(a) Order of Integration: $\int_a^b f(x)\ dx =$ _______________

(b) Zero Width Interval: $\int_a^a f(x)\ dx =$ _____

(c) Constant Multiple: $\int_a^b kf(x)\ dx =$ ______________

(d) Sum and Difference: $\int_a^b (f(x) \pm g(x))\ dx =$ ________________________

(e) Additivity: $\int_a^b f(x)\ dx + \int_b^c f(x)\ dx =$ _____________

(f) Max-Min Inequality: If f has maximum value max f and minimum value min f on $[a, b]$, then

____.

(g) Domination:

i. $f(x) \geq g(x)$ on $[a, b] \Rightarrow$ ________________________.

ii. $f(x) \geq 0$ on $[a, b] \Rightarrow$ ________________.

3 Area Under the Graph of a Nonnegative Function

3.1 *Definitions: Area Under a Curve as a Definite Integral*

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the ____________________ over $[a, b]$ is the __________ of f from a to b,

$$A = \underline{\hspace{3cm}}.$$

3.2 Equations

(a) $\displaystyle\int_a^b x\,dx = \underline{\hspace{4cm}}.$

(b) $\displaystyle\int_a^b c\,dx = \underline{\hspace{4cm}}.$

(c) $\displaystyle\int_a^b x^2\,dx = \underline{\hspace{4cm}}.$

Ex. 1 .... (example4, p268)

Compute $\displaystyle\int_0^b x\,dx$ and find the area under $y = x$ over the interval $[0, b]$, $b > 0$.

sol:

4 Average Value of a Continuous Function

4.1 *Definitions: Average or Mean Value of a Function*

> If f is integrable on $[a, b]$, then its average value (mean value) on $[a, b]$, is
>
> ________________________

4.2 The average value of a nonnegative continuous function f over an interval $[a, b]$ is ______________________________________.

✐ Ex. **2** .... (example5, p270)

Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

sol:

實習課練習 (EXERCISE 5.3)

☐ Express the limits as definite integrals.

1. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k^2 \Delta x_k$, where P is a partition of $[0, 2]$.

4. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\frac{1}{c_k}) \Delta x_k$, where P is a partition of $[1, 4]$.

6. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4 - c_k^2} \Delta x_k$, where P is a partition of $[0, 1]$.

☐ Evaluate the integrals.

29. $\int_1^{\sqrt{2}} x\ dx$

34. $\int_0^{0.3} s^2\ ds$

44. $\int_0^{\sqrt{2}} (t - \sqrt{2})\ dt$

45. $\int_2^1 (1 + \frac{z}{2})\ dz$

49. $\int_1^0 (3x^2 + x - 5)\ dx$

☐ Find the area of the region between the given curve and the x-axis on the interval $[0, b]$

52. $y = \pi x^2$.

54. $y = \frac{x}{2} + 1$.

☐ Find the average value over the given interval.

56. $f(x) = -\frac{x^2}{2}$ on $[0, 3]$.

61. $g(x) = |x| - 1$ on (a) $[-1, 1]$, (b) $[1, 3]$, (c) $[-1, 3]$.

THOMAS' CALCULUS (12/E)

5.4 The Fundamental Theorem of Calculus

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1 Mean Value Theorem for Definite Integrals

1.1 *Theorem 3: Mean Value Theorem for Definite Integrals*

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

______________________________.

Ex. 1 .... (example2, p275)

Show that if f is continuous on $[a, b]$, $a \neq b$, and if $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ at least once in $[a, b]$.

sol:

2 Fundamental Theorem, Part 1

2.1 *Theorem 4: The Fundamental Theorem of Calculus Part 1*

If f is continuous on $[a, b]$ then

(a) $F(x) =$ ________________ is continuous on $[a, b]$ and differentiable on (a, b),

(b) and its derivative is __________;

____________________________________.

✎ Ex. **2** .... (example2, p276)

Use the Fundamental Theorem to find dy/dx if

1. $y = \int_a^x (t^3+1)\ dt$

2. $y = \int_x^5 3t \sin t\ dt$

3. $y = \int_1^{x^2} \cos t\ dt$

3 Fundamental Theorem, Part 2 (The Evaluation Theorem)

3.1 *Theorem 4: The Fundamental Theorem of Calculus Part 2*

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

________________________________.

✎ Ex. **3** .... (example3, p278)

1. $\int_0^\pi \cos x\ dx =$

2. $\int_{-\pi/4}^{0} \sec x \tan x \; dx =$

3. $\int_{1}^{4} (\frac{3}{2}\sqrt{x} - \frac{4}{x^2}) \; dx =$

4 Total Area

4.1 To compute the ________________ bounded by the graph of a function and the x-axis requires more care when the function takes on both __________ and __________ values.

4.2 We must be careful to ________________________ into subintervals on which the function ___________________.

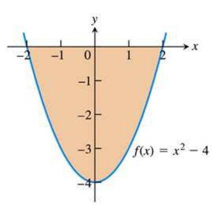
4.3 *Summary:*

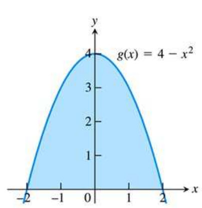
> To find the area between the graph of and the x-axis over the interval $[a, b]$, do the following: (a) Subdivide ____________________. (b) Integrate _____________________. (c) Add ______________________________.

Ex. 4 . . . . (example6, p280)

Figure 5.20 shows the graph of $f(x) = x^2 - 4$ and its mirror image $g(x) = 4 - x^2$ reflected across the x-axis. For each function, compute (a) the definite integral over the interval $[-2, 2]$ and (b) the area between the graph and the x-axis over $[-2, 2]$.

sol:





✎ Ex. **5** .... (`example6, p280`)

Let $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute (a) the definite integral of $f(x)$ over $[0, 2\pi]$ and (b) the area between the graph $f(x)$ and the x-axis over $[0, 2\pi]$.

sol:

✎ Ex. **6** .... (`example8, p281`)

Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$.

sol:

實習課練習 (EXERCISE 5.4)

☐ Evaluate the integrals.

1. $\displaystyle\int_{-2}^{0}(2x+5)\ dx$

8. $\displaystyle\int_{1}^{32} x^{-6/5}\ dx$

14. $\displaystyle\int_{-\pi/3}^{\pi/3}\frac{1-\cos 2t}{2}\ dt$

19. $\displaystyle\int_{1}^{-1}(r+1)^2\ dr$

24. $\displaystyle\int_{1}^{8}\frac{(x^{1/3}+1)(2-x^{2/3})}{x^{1/3}}\ dx$

27. $\displaystyle\int_{-4}^{4}|x|\ dx$

☐ Find the derivatives (a) by evaluating the integral and differentiating the result. (b) by differentiating the integral directly.

29. $\displaystyle\frac{d}{dx}\int_{0}^{\sqrt{x}}\cos t\ dt$

30. $\displaystyle\frac{d}{dx}\int_{1}^{\sin x} 3t^2\ dt$

☐ Find dy/dx.

33. $\displaystyle y=\int_{0}^{x}\sqrt{1+t^2}\ dt$

39. $\displaystyle y=\int_{0}^{\sin x}\frac{dt}{\sqrt{1-t^2}}\ dt,\ |x|<\frac{\pi}{2}$

☐ Find the total area between the region and the x-axis.

41. $y=-x^2-2x,\ -3\le x\le 2$

44. $y=x^{1/3}-x,\ -1\le x\le 8$

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5.5 Indefinite Integrals and the Substitution Rule

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1 The Power Rule in Integral Form

1.1 If u is a differentiable function of x and n is a rational number, $x \neq -1$, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = \underline{\qquad\qquad}.$$

1.2 If u is any differentiable function, then

$$\int u^n \, du = \underline{\qquad\qquad\qquad}, \quad (n \neq -1, n \text{ rational})$$

Ex. 1 .... (example1, p285)

Find the integral $\int (x^3+x)^5(3x^2+1)\, dx$

sol:

Ex. 2 .... (example2, p285)

Find the integral $\int \sqrt{2x+1}\, dx$

sol:

2 Substitution: Running the Chain Rule Backwards

2.1 *Theorem 6: The Substitution Rule*

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

______________________________.

(a) Substitute __________ and ____________ to obtain the integral ____________.

(b) Integrate with respect to _____.

(c) Replace _____ by _______ in the result.

Ex. 3 .... (example3, p287)

Find $\int \sec^2(5t+1)\cdot 5\ dt$

sol:

Ex. 4 .... (example4, p287)

Find $\int \cos(7\theta+3)\ d\theta$

sol:

Ex. 5 .... (example5, p287)

Evaluate $\int x^2 \sin(x^3)\ dx$

sol:

✎ Ex. **6** .... (example6, p288)

Evaluate $\int x\sqrt{2x+1}\,dx$.

sol:

✎ Ex. **7** .... (example7, p288)

Evaluate $\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz$

sol:

3 The Integrals of $\sin^2 x$ and $\cos^2 x$

✎ Ex. **8** .... (example7, p7)373

(a) $\int \sin^2 x\ dx$

(b) $\int \cos^2 x\ dx$

實習課練習 (EXERCISE 5.5)

8. $\int x\sin(2x^2)\ dx$

11. $\int \frac{9r^2}{\sqrt{1-r^3}}\ dr$

13. $\int \sqrt{x}\sin^2(x^{3/2}-1)\ dx$

16. $\int \frac{dx}{\sqrt{5x+8}}$

19. $\int \theta\sqrt[4]{1-\theta^2}\ d\theta$

21. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\ dx$

25. $\int \sin^5\frac{x}{3}\cos\frac{x}{3}\ dx$

31. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\ dt$

38. $\int \sqrt{\frac{x-1}{x^5}}\ dx$

46. $\int (x+5)(x-5)^{1/3}\ ds$

48. $\int 3x^5\sqrt{x^3+1}\ dx$

50. $\int \frac{x}{(x-4)^3}\ dx$

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5.6 Substitution and Area Between Curves

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1 Substitution Formula

1.1 *Theorem 7: Substitution in Definite Integrals*

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then

____

Ex. 1 .... (example1, p292)

Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

sol(1):

sol(2):

Ex. 2 .... (example2, p293)

Evaluate $\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta \; d\theta$.

sol:

2 Definite Integrals of Symmetric Functions

2.1 (a) f is a even function if ________________.

(b) f is a odd function if ________________.

2.2 *Theorem 7*

Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\; dx =$ ________________.

(b) If f is odd, then $\int_{-a}^{a} f(x)\; dx =$______.

Ex. 3 .... (example3, p294)

Evaluate $\int_{-2}^{2} (x^4 - 4x^2 + 6)\; dx$.

sol:

3 Areas Between Curves

3.1 *Definition: Area Between Curves*

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of ____________ from a to b:

$$A = \underline{\hspace{4cm}}.$$

✎ Ex. 4 .... (example4, p295)

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

sol:

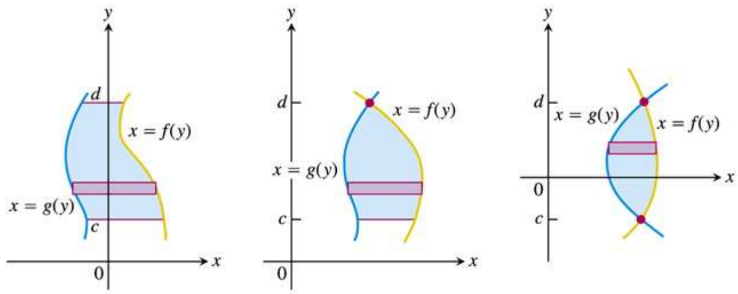
✎ Ex. 5 .... (example5, p295)

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

sol:

4 Integration with Respect to y

4.1 If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x. (圖示如下)



4.2 Use the formula $A =$ ____________________, where f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

Ex. 6 . . . . (`example6, p296`)

Find the area of the region in Example 5 by integrating with respect to y.

sol:

實習課練習 (EXERCISE 5.6)

☐ Evaluate the integrals.

2. $\int_{-1}^{1} r\sqrt{1-r^2}\ dr$

4. $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\ dx$

8. $\int_{1}^{4} \frac{10\sqrt{v}}{(1+v^{3/2})^2}\ dv$

15. $\int_{0}^{1} \sqrt{t^5+2t}(5t^4+2)\ dt$

24. $\int_{-1}^{-1/2} t^{-2}\sin^2(1+\frac{1}{t})\ dt$

☐ Find the areas of the regions enclosed by the lines and curves.

48. $y = x\sqrt{a^2-x^2}, a > 0$ and $y = 0$.

50. $y = |x^2-4|$, and $y = (x^2/2)+4$.

54. $x - y^2 = 0$, and $x + 2y^2 = 3$.

61. $x + 4y^2 = 4$, and $x + y^4 = 1$, for $x \geq 0$.

63. $y = 2\sin x$, and $y = \sin 2x$, $0 \leq x \leq \pi$.

73. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

76. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to (a) x, (b) y.

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6.1 Volumes Using Cross-Sections

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1 Volumes of Solids

1.1 A cross-section of a solid S is the _______________ formed by _______________ S with a __________ (Figure 6. 1).

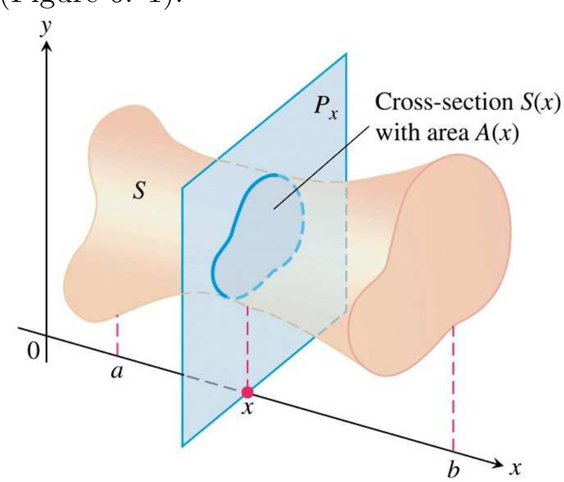


FIGURE 6.1 A cross-section $S(x)$ of the solid S formed by intersecting S with a plane P_x perpendicular to the x-axis through the point x in the interval $[a, b]$.

1.2 Three different methods for obtaining the cross-sections appropriate to finding the volume of a particular solid: the method of __________, the ________ method, and the __________ method.

1.3 (Figure 6.2) If the cylindrical solid has a known _______________ and ____________, then the volume of the cylindrical solid is

$$\text{Volume} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$

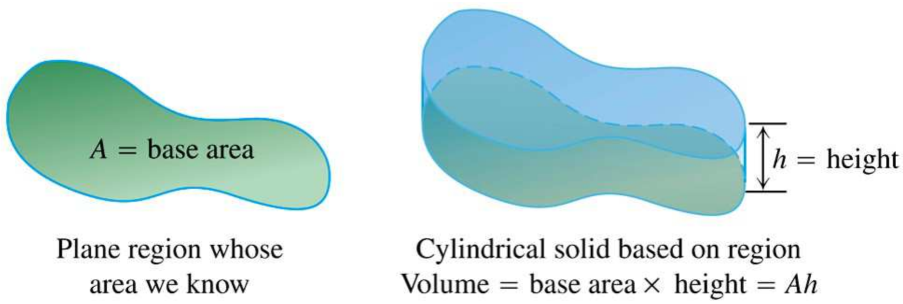


FIGURE 6.2 The volume of a cylindrical solid is always defined to be its base area times its height.

1.4 If the cross-section of the solid S at each point x in the interval ________ is a ______________ of _____________, and A is a continuous function of x, we can define and calculate the volume of the solid S as the _________________________.

2 Slicing by Parallel Planes

2.1 Partition ________ into subintervals of width (length) _______ and slice the solid by planes ________________ to the x-axis at the partition points ____________________________. The planes _______, perpendicular to the x-axis at the partition points, slice S into thin __________ (Figure 6.3).

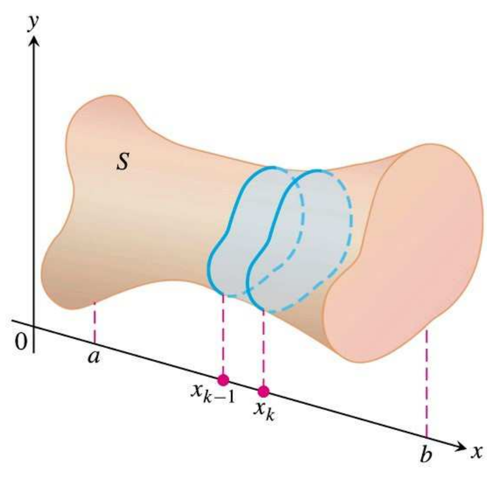


FIGURE 6.3 A typical thin slab in the solid S.

2.2 Approximate the slab between the plane at ________ and the plane at _______ by a __________________ with base area _________ and height __________________

(Figure 6.4).

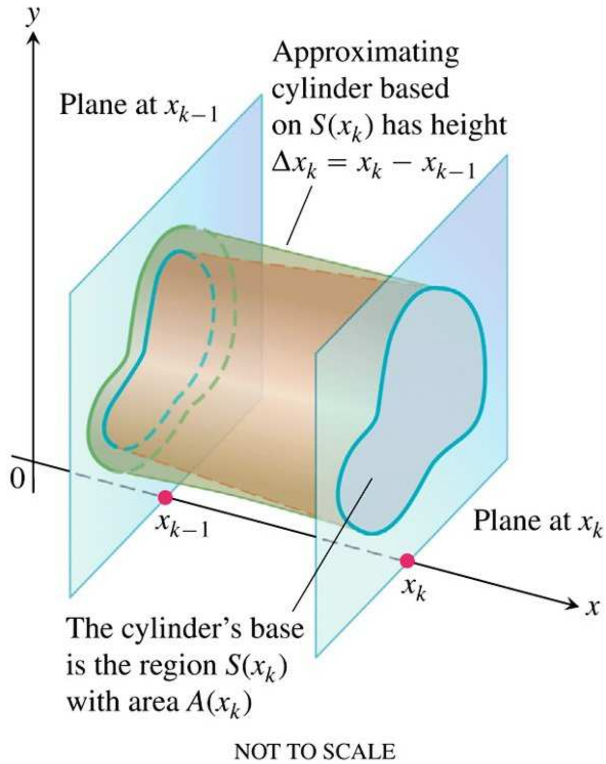


FIGURE 6.4 The solid thin slab in Figure 6.3 is shown enlarged here. It is approximated by the cylindrical solid with base $S(x_k)$ having area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$.

2.3 The volume V_k of this cylindrical solid is ____________, which is approximately the same volume as that of the slab:

$$\text{Volume of the } k\text{th slab} \underline{\hspace{4cm}}.$$

2.4 The volume V of the entire solid S is therefore approximated by the sum of these cylindrical volumes,

$$V \approx \underline{\hspace{2.5cm}} = \underline{\hspace{4cm}}.$$

This is a Riemann sum for the function $A(x)$ on $[a, b]$.

2.5 Taking a partition of $[a, b]$ into _____ subintervals with __________ gives

$$\underline{\hspace{5cm}} = \underline{\hspace{3cm}}.$$

2.6 **Definition: The Volume of a Solid**

The volume of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \underline{\hspace{3cm}}.$$

2.7 **Calculating the Volume of a Solid**

(a) Sketch the solid and a typical ________.

(b) Find a formula for ______, the ______ of a typical cross-section.

(c) Find the ______ of integration.

(d) Integrate ______ to find the volume.

Ex. 1 .... (example1, p309)

A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

sol:

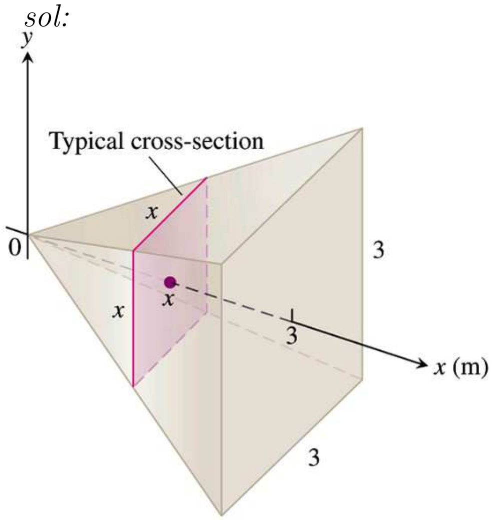


FIGURE 6.5 The cross-sections of the pyramid in Example 1 are squares.

✎ Ex. **2** .... (`example2, p310`)

A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

sol:

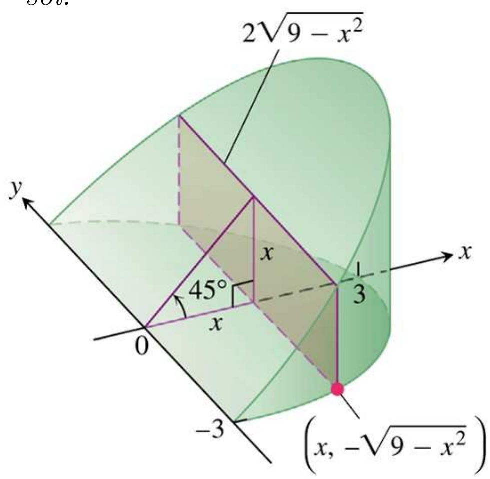


FIGURE 6.6 The wedge of Example 2, sliced perpendicular to the x-axis. The cross-sections are rectangles.

3 Solids of Revolution: The Disk Method

3.1 The solid generated by ____________ (or ____________) a plane region about an ________ in its plane is called ______________________.

3.2 To find the volume of a solid like the one shown in Figure 6.8, we need only observe that the cross-sectional area _________ is the area of _______________________, the distance of the planar region's boundary from the axis of revolution. The area is then

$$A(x) = \underline{\qquad\qquad} = \underline{\qquad\qquad}$$

3.3 **Volume by Disks for Rotation About the x-axis**

$$V = \underline{\qquad\qquad\qquad} = \underline{\qquad\qquad\qquad\qquad}.$$

3.4 To find the volume of a solid generated by revolving a region between the __________ and a curve __________, __________, about the y-axis, we use the same method with x replaced by y, the circular cross-section is

$$A(y) = \underline{\qquad\qquad} = \underline{\qquad\qquad}$$

3.5 **Volume by Disks for Rotation About the y-axis**

$$V = \underline{\qquad\qquad\qquad} = \underline{\qquad\qquad\qquad\qquad}.$$

✎ Ex. **3** . . . . (`example4, p311`)

The region between the curve $y = \sqrt{x}, 0 \leq x \leq 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

sol:

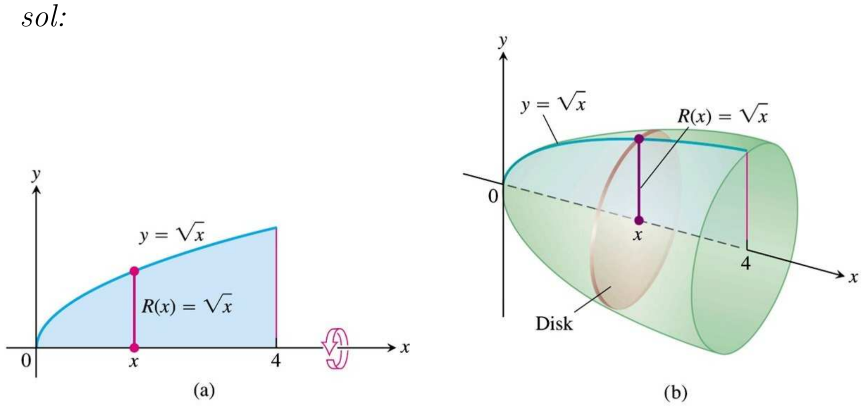


FIGURE 6.8 The region (a) and solid of revolution (b) in Example 4.

✎ Ex. 4 .... (example6, p311)

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

sol:

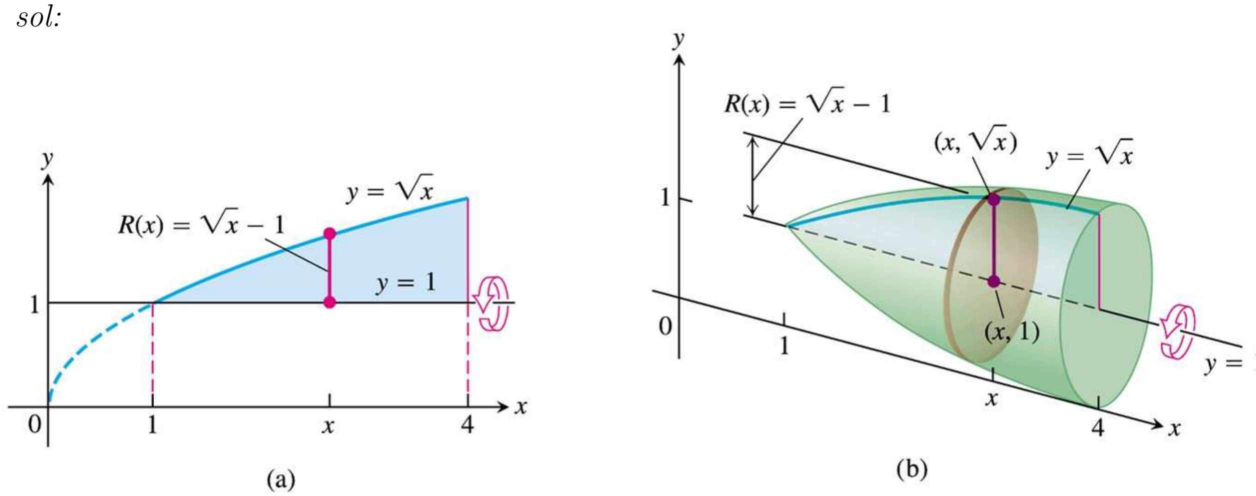


FIGURE 6.10 The region (a) and solid of revolution (b) in Example 6.

✎ Ex. 5 .... (example7, p313)

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y, 1 \leq y \leq 4$, about the y-axis.

sol:

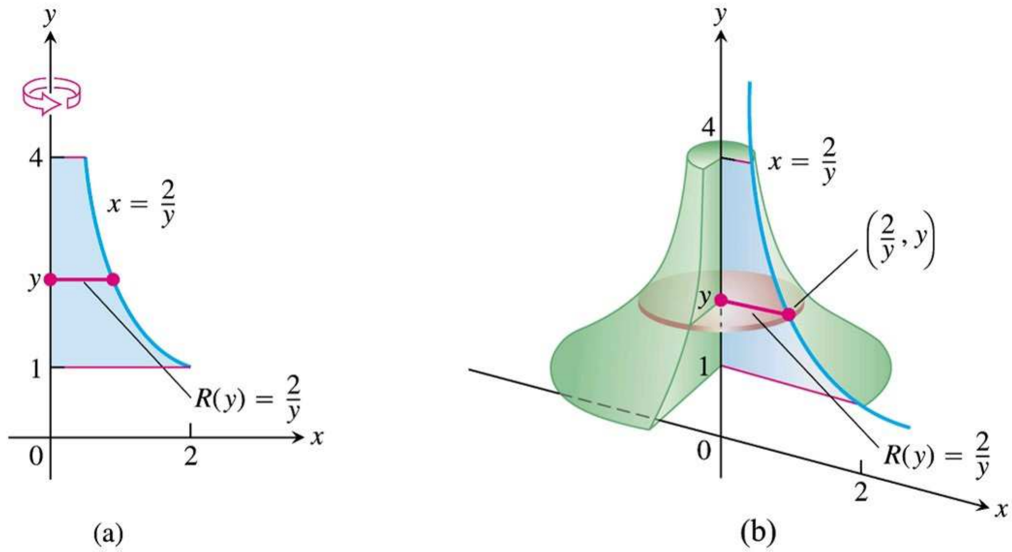


FIGURE 6.11 The region (a) and part of the solid of revolution (b) in Example 7.

✎ Ex. 6 .... (example8, p313)

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

sol:

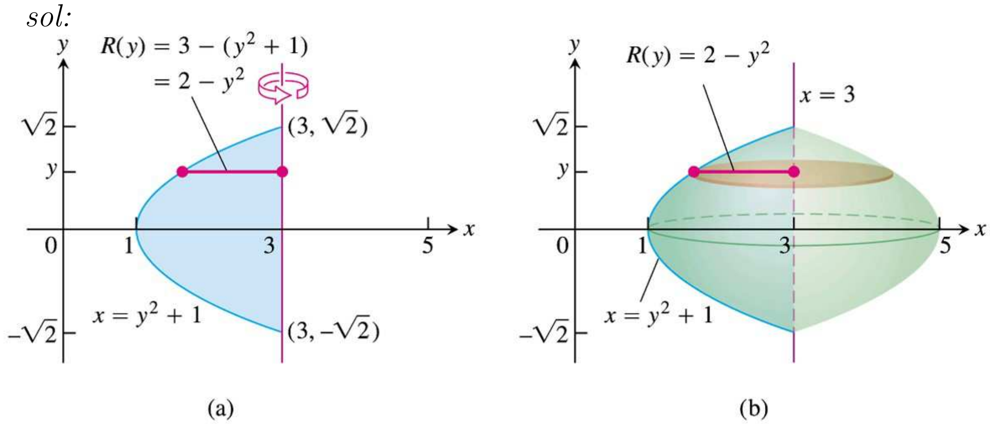


FIGURE 6.12 The region (a) and solid of revolution (b) in Example 8.

4 Solids of Revolution: The Washer Method

4.1 If the region we revolve to generate a solid does not ____________ or ______________ ______________, the solid has a ________ in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are ___________ instead of disks. The dimensions of a typical washer are: ____________________, ___________________.

4.2 The washer's area is

$$A(x) = \underline{\hspace{4cm}} = \underline{\hspace{4cm}}.$$

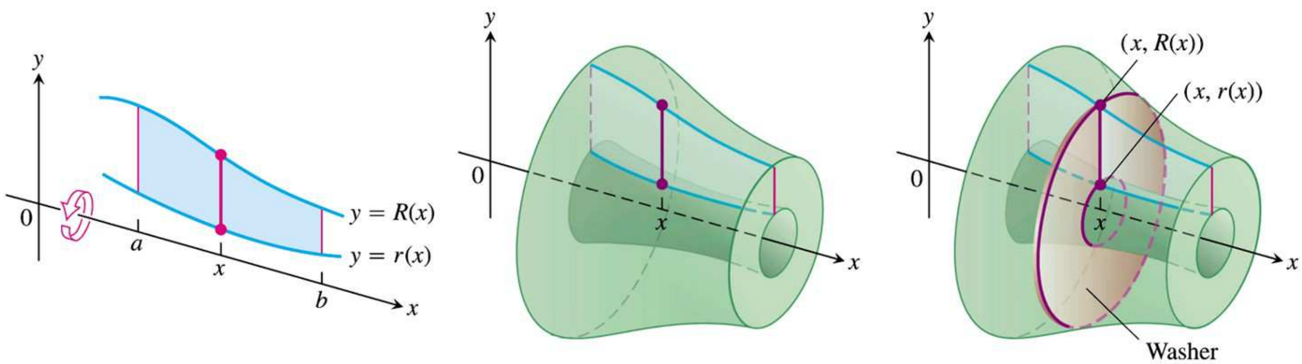


FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.

4.3 **Volume by Washers for Rotation About the x-axis**

$$V = \underline{\hspace{3cm}} = \underline{\hspace{5cm}}.$$

✎ Ex. 7 .... (example9, p314)

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

sol:

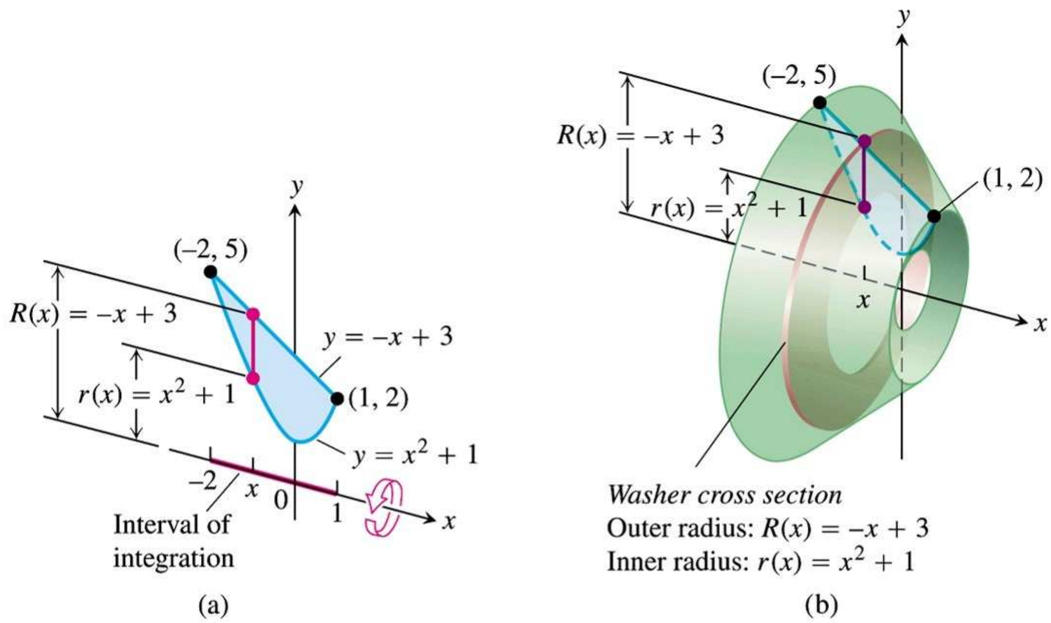


FIGURE 6.14 (a) The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. (b) When the region is revolved about the x-axis, the line segment generates a washer.

✎ Ex. **8** .... (example10, p315)

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

sol:

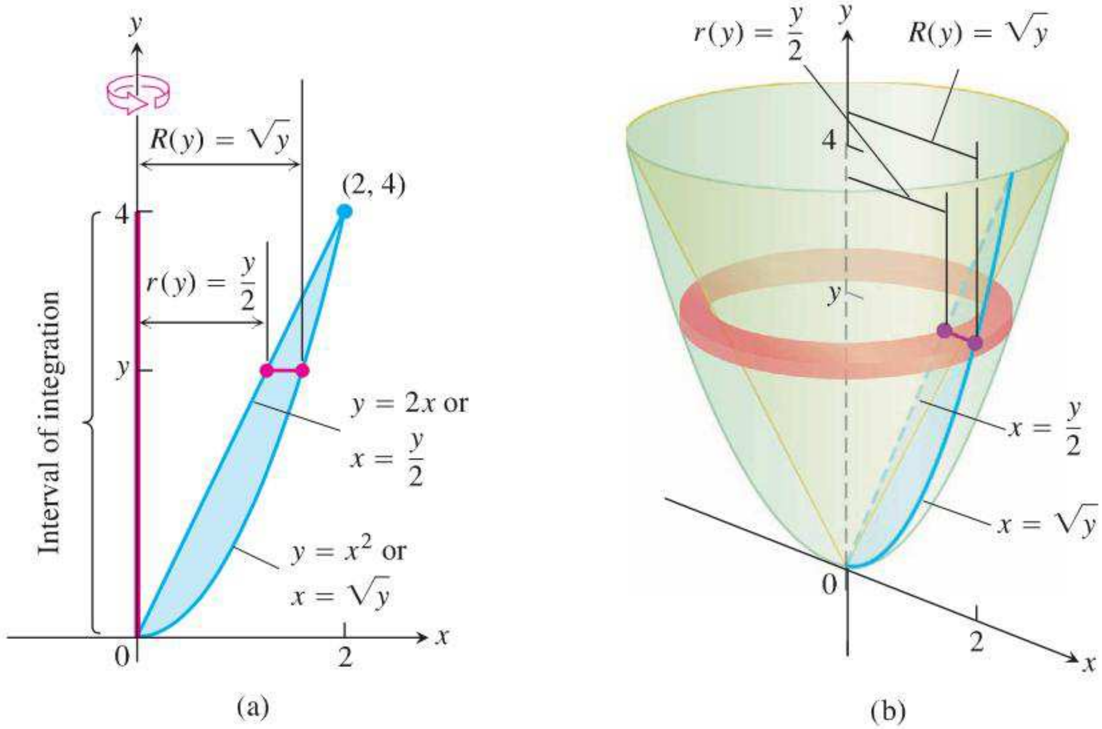


FIGURE 6.15 (a) The region being rotated about the y-axis, the washer radii, and limits of integration in Example 10. (b) The washer swept out by the line segment in part (a).

實習課練習 (EXERCISE 6.1)

In Exercises 1-10, Find the volumes of the solids.

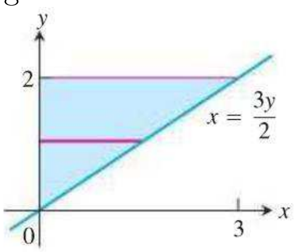
3. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are squares whose bases run from the semi- circle $y = -\sqrt{1-x^2}$ to the semicircle $y = \sqrt{1-x^2}$.

9. The base of a solid is the region bounded by the graphs of $y = \sqrt{x}$ and $y = x/2$. The cross-sections perpendicular to the x-axis are

a. isosceles triangles of height 6.

b. semi-circles with diameters running across the base of the solid.

16. Find the volume of the solid generated by revolving the shaded region about the given axis. About the y-axis.

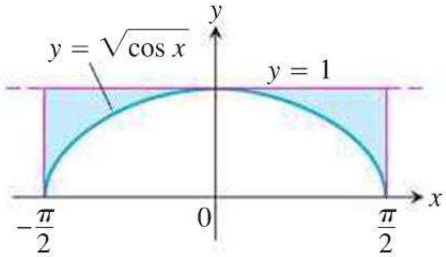


22. Find the volumes of the solids generated by revolving the regions bounded by the lines and curves about the x-axis. $y = x - x^2, y = 0$.

26. The region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2\sin x, 0 \le x \le \pi/2$, and on the left by the y-axis, about the line $y = 2$.

27. Find the volumes of the solids generated by revolving the regions bounded by the lines and curves about the y-axis. $x = \sqrt{2y}/(y^2+1), x = 0, y = 1$.

33. Find the volumes of the solids generated by revolving the shaded regions about the indicated axes.



47. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about

a. the x-axis. **b.** the y-axis. **c.** the line $y = 2$. **d.** the line $x = 4$.

50. By integration, find the volume of the solid generated by revolving the triangular region with vertices $(0, 0), (b, 0), (0, h)$ about

a. the x-axis. **b.** the y-axis.

THOMAS' CALCULUS (12/E)

7.1 Inverse Function and Their Derivatives

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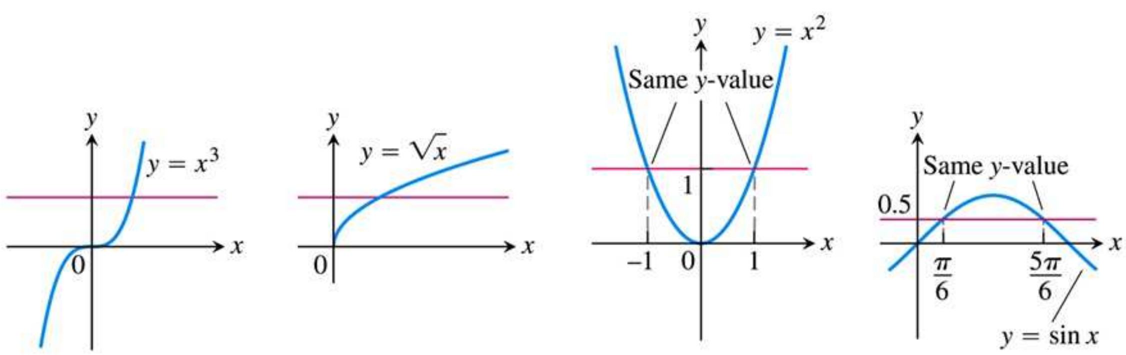
1 One-to-One Functions and Inverse Functions

1.1 *Definitions: One-to-One Function*

> A function $f(x)$ is one-to-one on a domain D if ______________ whenever __________ in D.

1.2 One-to-one: (a) $y = x^3$, (b) $y = \sqrt{x}$. (圖示如下)

Not one-to-one: (c) $y = x^2$, (d) $y = \sin x$. (圖示如下)



1.3 A function $y = f(x)$ is one-to-one if and only if its graph interests each _______________ at most ________.

1.4 *Definitions: Inverse Function*

> Suppose that f is a one-to-one function on a domain D with range R. The inverse function _______ is defined by
>
> ____________ if __________
>
> The __________ of f^{-1} is ______ and the ________ of f^{-1} is ______.

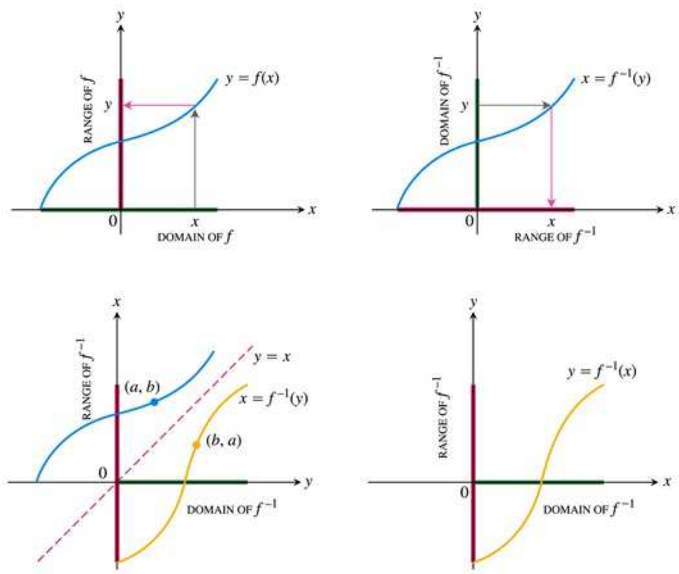
1.5 $(f^{-1} \cdot f)(x) =$ ______ for all x in the domain of f.

1.6 $(f \cdot f^{-1})(y) =$ ______ for all y in the domain of f^{-1}.

1.7 Only a one-to-one function can have an __________.

2 Finding Inverses

2.1 Determining the graph of $y = f^{-1}(x)$ from the graph of $y = f(x)$. (圖示如下)



2.2 Pass from f to f^{-1}.

(a) Solve the equation ____________ for x. This gives a formula ______________ where x is expressed as a function of y.

(b) Interchange ___________, obtaining a formula ______________ where f^{-1} is expressed in the conventional format with x as the _______________ variable and y as the _____________________.

✎ Ex. 1 .... (example3, p364)

Find the inverse of $y = \dfrac{1}{2}x + 1$, expressed as a function of x.

sol:

✎ Ex. **2** .... (example4, p364)

Find the inverse of the function $y = x^2,\ x \geq 0$, expressed as a function of x.

sol:

3 Derivatives of Inverses of Differentiable Functions

3.1 $f(x) = (1/2)x + 1$ and $f^{-1}(x) =$__________.

$$\frac{d}{dx} f(x) = \underline{\hspace{4cm}}$$

$$\frac{d}{dx} f^{-1}(x) = \underline{\hspace{4cm}}$$

3.2 *Theorem 1: The Derivative Rule for Inverses*

(a) If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is ________________ at every point in its domain.

(b) The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the ______________ of f' the value of at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \underline{\hspace{3cm}} \quad \text{or} \quad \left.\frac{d}{dx} f^{-1}\right|_{x=b} = \underline{\hspace{3cm}}$$

3.3 When $y = f(x)$ is differentiable at $x = a$ and we change x by a small amount dx, the corresponding change in y is approximately ________________. This means that y changes about __________ times as fast as x when $x = a$ and that x changes about ____________ times as fast as y when $y = b$.

3.4 It is reasonable that the derivative of f^{-1} at b is the ______________ of the derivative of f at a.

✎ Ex. **3** .... (example5, p366)

Apply The Derivative Rule for Inverse Theorem to the function $f(x) = x^2$, $x \geq 0$.

sol:

✎ Ex. 4 .... (example6, p366)

Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

sol:

實習課練習 (EXERCISE 7.1)

21. Let $f(x) = x^3 - 1$. Find a formula for f^{-1}.

22. Let $f(x) = x^2 - 2x + 1,\ x \geq 1$. Find a formula for f^{-1}.

33. Let $f(x) = x^2 - 2x,\ x \leq 1$. Find f^{-1} and identify the domain and range of f^{-1}.

37. Let $f(x) = 5 - 4x,\ a = 1/2$. Find $f^{-1}(x)$. Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$.

38. Let $f(x) = 2x^2,\ x \geq 0,\ a = 5$. Find $f^{-1}(x)$. Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$.

41. Let $f(x) = x^3 - 3x^2 - 1,\ x \geq 2$. Find the value of df^{-1}/dx at the point $x = -1 = f(3)$.

THOMAS' CALCULUS (12/E)

7.2 Natural Logarithms

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1 Definition of the Natural Logarithm Function

1.1 The natural logarithm of a positive number x, written as ________, is the value of an integral.

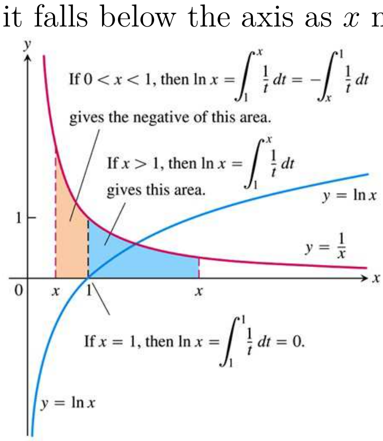
1.2 *Definitions: The Natural Logarithm Function*

$$\ln x = \underline{\qquad\qquad\qquad}, \quad x > 0$$

1.3 If $x > 1$, then $\ln x$ is the area under the curve __________ from ________ to ________.

1.4 For $0 < x < 1$, $\ln x$ gives the ____________________ under the curve from _____ to _____.

1.5 The graph of $y = \ln x$ and its relation to the function $y = 1/x,\ x > 0$.
The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the axis as x moves from 1 to the left. (圖示如下)



1.6 *Definitions: The Number e*

> The number e is that number in the domain of the natural logarithm satisfying ____________.

1.7 $\ln 1 =$ ______________ .

1.8 Geometrically, the number e corresponds to the point on the x-axis for which the area under the graph of __________ and above the interval ________ is the exact area of the unit square.

2 The Derivative of $y = \ln x$

2.1 $\dfrac{d}{dx}\ln x =$ ______________ $=$ ______

2.2 $y = \ln u$

$$\frac{d}{dx}\ln u = \underline{\hspace{4cm}} = \underline{\hspace{2.5cm}}, \quad u > 0$$

Ex. 1 .... (`example1, p371`)

1. $\dfrac{d}{dx}\ln 2x =$
2. $\dfrac{d}{dx}\ln(x^2+3) =$
3. $\dfrac{d}{dx}\ln ax =$

3 Properties of Logarithms

3.1 *Theorem 2: Properties of Logarithms*

> For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:
>
> (a) Product Rule: ____________________
>
> (b) Quotient Rule: ____________________
>
> (c) Reciprocal Rule: ________________
>
> (d) Power Rule: ________________

Ex. **2** .... (example2, p372)

1. $\ln 4 + \ln \sin x =$
2. $\ln \dfrac{x+1}{2x-3} =$
3. $\ln \sec x =$
4. $\ln \sqrt[3]{x+1} =$

4 The Integral $\int (1/u)du$

4.1 If u is a differentiable function that is never zero,

$$\int \frac{1}{u}\, du = \underline{\hspace{3cm}}.$$

4.2 If $u = f(x)$, then $du =$ __________ and

$$\int \frac{1}{u}\, du = \frac{\hspace{3cm}}{\underline{\hspace{3cm}}} = \underline{\hspace{3.5cm}}.$$

Ex. **3** .... (example3, p374)

$$\int_0^2 \frac{2x}{x^2-5}\, dx$$

sol:

5 The Integral of $\tan x$, $\cot x$, $\sec x$ and $\csc x$

5.1 $\int \tan u\, du = \underline{\hspace{3.5cm}}$

Proof:

5.2 $\int \cot u \; du =$ ______________

Proof:

5.3 $\int \sec u \; du =$ ____________________

Proof:

5.4 $\int \csc u \; du =$ ______________________

Proof:

✎ Ex. 4 .... (example4, p375)

$$\int_0^{\pi/6} \tan 2x \; dx$$

sol:

6 Logarithmic Differentiation

✎ Ex. **5** .... (example5, p375)

Find dy/dx if $y = \dfrac{(x^2+1)(x+3)^{1/2}}{x-1}, \quad x > 1.$

sol:

實習課練習 (EXERCISE 7.2)

2. Express the following logarithms in terms of $\ln 5$ and $\ln 7$. (a) $\ln(1/125)$, (b) $\ln 9.8$, (c) $\ln 7\sqrt{7}$, (d) $\ln 1225$, (e) $\ln 0.056$, (f) $(\ln 35 + \ln(1/7))/(\ln 25)$.

3. Simplify the expressions: (a) $\ln \sin\theta - \ln(\frac{\sin\theta}{5})$, (b) $\ln(3x^2 - 9x) + \ln(\frac{1}{3x})$, (c) $\frac{1}{2}\ln(4t^4) - \ln 2$.

☐ Find the derivative of y with respect to x, or t, as appropriate.

8. $y = \ln(t^3/2)$

15. $y = t(\ln t)^2$

22. $y = \dfrac{x \ln x}{1 + \ln x}$

35. $y = \displaystyle\int_{x^2/2}^{x^2} \ln\sqrt{t}\ dt$

☐ Evaluate the integrals:

38. $\displaystyle\int_{-1}^{0} \frac{3}{3x - 2}\ dx$

44. $\displaystyle\int_{2}^{4} \frac{dx}{x \ln x}$

45. $\displaystyle\int_{2}^{4} \frac{dx}{x(\ln x)^2}$

52. $\displaystyle\int_{0}^{\pi/12} 6\tan 3x\ dx$

53. $\displaystyle\int \frac{dx}{2\sqrt{x} + 2x}$

☐ Use logarithmic differentiaton to find the derivative of y with respect to the given independent variable.

55. $y = \sqrt{x(x+1)}$

62. $y = \dfrac{1}{t(t+1)(t+2)}$

67. $y = \sqrt[3]{\dfrac{x(x-2)}{x^2+1}}$

THOMAS' CALCULUS (12/E)

7.3 Exponential Function

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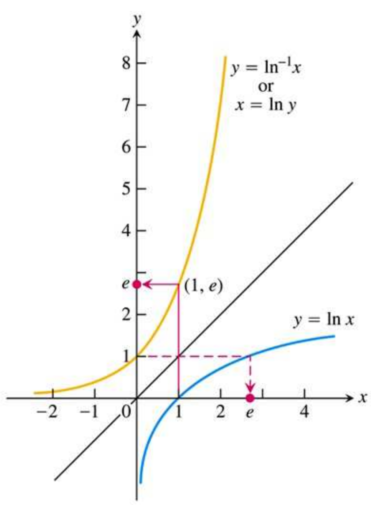
1 The Inverse of $\ln x$ and e^x

1.1 The function $\ln x$, being an increasing function of x with domain $(0, \infty)$ and range $(-\infty, \infty)$, has an inverse $\ln^{-1} x$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. (圖示如下)



1.2 $\ln(e) =$______, $e =$ ____________ $=$ ____________.

1.3 $e \doteq$ ________________________.

1.4 *Definitions: The Natural Exponential Function*

For every real number x, $e^x =$ ____________ $=$ ____________.

1.5 $\ln e^r =$____________________ $\Rightarrow$ $e^r =$__________________.

1.6 Inverse Equation for e^x and $\ln x$

$$e^{lnx} = \_\_\_\_\_\_, \quad \forall x > 0$$

$$\ln(e^x) = \_\_\_\_\_\_, \quad \forall x$$

Ex. 1 .... (`example1, p378`)

Solve the equation $e^{2x-6} = 4$ for x.

sol:

2 The Derivative and Integral of e^x

2.1 Let $f(x) = \ln x$ and $y = e^x =$ __________________. Then $\dfrac{dy}{dx} =$ _______

____

2.2 If u is any differentiable function of x, then $\dfrac{d}{dx}\, e^u =$ _________

2.3 $\displaystyle\int e^u \, du =$ ___________

Ex. 2 .... (`example2, p379`)

(a) $\dfrac{d}{dx}\,(5e^x) =$

(b) $\dfrac{d}{dx}\, e^{-x} =$

(c) $\dfrac{d}{dx}\, e^{\sin x} =$

(d) $\dfrac{d}{dx}\,(e^{\sqrt{3x+1}}) =$

Ex. **3** . . . . (`example3, p379`)

(a) $\int_0^{\ln 2} e^{3x}\, dx =$

(b) $\int_0^{\pi/2} e^{\sin x} \cos x\, dx =$

3 Laws of Exponents

3.1 *Definitions: General Exponential Functions*

For any numbers $a > 0$ and x, the exponential function with base a is ______________.

3.2 *Theorem 3: Laws of Exponents for e^x*

For all numbers x, x_1 and x_2, the natural exponential e^x obeys the following laws:

(a) $e^{x_1} \cdot e^{x_2} =$__________

(b) $e^{-1} =$_________

(c) $e^{x_1}/e^{x_2} =$__________

(d) $((e^{x_1})^{x_2}) =$__________________

Proof of Law (a):

Ex. **4** . . . . (`example4, p382`) Differentiate $f(x) = x^x,\ x > 0$.

sol:

4 The Number e Expressed as a Limit

4.1 *Theorem 4: The Number e as a Limit*

The number e can be calculated as the limit: $e =$ ________________.

Proof:

4.2 *Power Rule (General Form)*

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and $\frac{d}{dx}u^n =$ ____________.

4.3 Examples:

(a) $\frac{d}{dx}x^{\sqrt{2}} =$

(b) $(2 + \sin 3x)^{\pi} =$

5 The Derivative of a^u

5.1 If $a > 0$, then $\frac{d}{dx}a^x =$__________.

Proof:

5.2 If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and $\frac{d}{dx}a^u =$ ____________.

5.3 If $a \neq 1$, $\int a^u \, du =$ __________.

Proof:

Ex. **5** .... (example5, p383)

(a) $\dfrac{d}{dx}\, 3^x =$

(b) $\dfrac{d}{dx}\, 3^{-x} =$

(c) $\dfrac{d}{dx}\, 3^{\sin x} =$

(d) $\displaystyle\int 2^x \, dx =$

(e) $\displaystyle\int 2^{\sin x} \cos x \, dx =$

6 Logarithms with Base a

6.1 *Definitions:* $\log_a x$

> For any positive number $a \neq 1$, ________ is the inverse function of a^x.

6.2 Inverse Equations for a^x and $log_a x$

$a^{\log_a x} =$ ______, $\quad x > 0$

$\log_a(a^x) =$ ______, $\forall x$

6.3 $\log_a x =$ ________

Proof:

6.4 Derivatives and Integrals Involving $\log_a x$:

$$\frac{d}{dx}(\log_a u) = \underline{\hspace{4cm}}$$

Ex. **6** .... (`example6, p385`)

(a) $\dfrac{d}{dx}\log_{10}(3x+1) =$

(b) $\displaystyle\int \frac{\log_2 x}{x}\, dx =$

實習課練習 (EXERCISE 7.3)

☐ Solve for t.

2. (a) $e^{-0.01t} = 1000$, (b) $e^{kt} = \frac{1}{10}$, (c) $e^{(\ln 2)x} = 1/2$.

4. $e^{x^2} e^{2x+1} = e^t$

☐ Find the derivative of y with respect to x, t or θ, as appropriate.

9. $y = xe^x - e^x$

14. $y = \ln(3\theta e^{-\theta})$

20. $y = \ln(\dfrac{\sqrt{\theta}}{1+\sqrt{\theta}})$

24. $y \displaystyle\int_{e^4\sqrt{x}}^{e^{2x}} \ln t \; dt$

25. $\ln y = e^y \sin x$

28. $\tan y = e^x + \ln x$

☐ Evaluate the integrals.

33. $\displaystyle\int 8e^{x+1} \; dx$

41. $\displaystyle\int \frac{e^{1/x}}{x^2} \; dx$

43. $\displaystyle\int_0^{\pi/4} (1 + e^{\tan\theta}) \sec^2\theta \; d\theta$

49. $\displaystyle\int \frac{e^r}{1+e^r} \; dr$

50. $\displaystyle\int \frac{dx}{1+e^x}$

☐ Find the derivative of y with respect to the given independent variable.

57. $y = 5^{\sqrt{s}}$

72. $y = \log_3 r \cdot \log_9 r$

81. $y = \log_2(8t^{\ln 2})$

☐ Evaluate the integrals.

85. $\displaystyle\int_0^1 2^{-\theta}\ d\theta$

91. $\displaystyle\int_2^4 x^{2x}(1+\ln x)\ dx$

96. $\displaystyle\int_1^e x^{(\ln 2-1)}\ dx$

99. $\displaystyle\int_1^4 \frac{\ln 2 \log_2 x}{x}\ dx$

106. $\displaystyle\int \frac{dx}{x(\log_8 x)^2}$

107. $\displaystyle\int_1^{\ln x} \frac{1}{t}\ dt, \quad x > 1$

☐ Find the derivative of y with respect to the given independent variable.

111. $y = x + 1^x$

116. $y = x^{\sin x}$

118. $y = (\ln x)^{\ln x}$

THOMAS' CALCULUS (12/E)

7.5 Indeterminate Forms and L'Hopital's Rule

開課班級: 資訊 1/電機 1/智財學程微積分
授課教師: 吳漢銘 (國立臺北大學統計學系副教授)
教學網站: http://www.hmwu.idv.tw

1 Indeterminate Form 0/0

1.1 If the continuous functions $f(x)$ and $g(x)$ are both ________ at $x = a$, then ____________ cannot be found by substituting $x = a$.

1.2 The substitution produces ________, a meaningless expression, which we cannot evaluate. We use ________ as a notation for an expression known as an ____________________.

1.3 *Theorem: L'Hopital's Rule (First Form)*

> Suppose that ________________, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then
> $$\lim_{x \to a} \frac{f(x)}{g(x)} = \underline{\qquad\qquad}.$$

Proof:

1.4 *Theorem: L'Hopital's Rule (Stronger Form)*

> Suppose that ________________, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then
> $$\lim_{x \to a} \frac{f(x)}{g(x)} = \underline{\qquad\qquad},$$
> assuming that the limit on the right side exists.

1.5 *Using L'Hopital's Rule*

To find $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ by L'Hopital's Rule,

(a) continue to differentiate f and g, so long as we still get the form ________ at $x = a$.

(b) But as soon as one or the other of these derivatives is different from ________ at $x = a$ we stop differentiating.

(c) L'Hopital's Rule does not apply when either the ______________ or ________________ has a finite ____________ limit.

✎ Ex. **1** .... (`example1, p397`)

(a) $\lim_{x\to 0} \dfrac{3x - \sin x}{x} =$

(b) $\lim_{x\to 0} \dfrac{\sqrt{1+x} - 1}{x} =$

(c) $\lim_{x\to 0} \dfrac{\sqrt{1+x} - 1 - x/2}{x^2} =$

(d) $\lim_{x\to 0} \dfrac{x - \sin x}{x^3} =$

Ex. 2 .... (example2, p398)

Find $\lim_{x \to 0} \frac{1 - \cos x}{x + x^2}$.

sol:

Ex. 3 .... (example3, p398)

(a) $\lim_{x \to 0^+} \frac{\sin x}{x^2} =$

(b) $\lim_{x \to 0^-} \frac{\sin x}{x^2} =$

2 Indeterminate Form $\infty/\infty, \infty \cdot 0, \infty - \infty$

2.1 L'Hopital's Rule applies to the indeterminate form ________.

2.2 If ______________ and ______________ as $x \to a$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \underline{\hspace{3cm}}$$

provided the limit on the right exists.

2.3 In the notation $x \to a$ may be either ________ or __________.

2.4 Moreover $x \to a$ may be replaced by the one-sided limits __________ or __________.

Ex. 4 .... (example4, p398)

(a) $\lim_{x \to \pi/2} \frac{\sec x}{1 + \tan x} =$

(b) $\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}} =$

(c) $\lim_{x\to\infty} \frac{e^x}{x^2} =$

Ex. 5 .... (`example5, p399`)

(a) Find $\lim_{x\to\infty}(x \sin \frac{1}{x})$.

(b) Find $\lim_{x\to 0^+}(\sqrt{x}\ln x)$.

sol:

Ex. 6 .... (`example6, p399`)

Find $\lim_{x\to 0}(\frac{1}{\sin x} - \frac{1}{x})$.

sol:

3 Indeterminate Powers

3.1 If $\lim_{x\to a} \ln f(x) = L$, then $\lim_{x\to a} f(x) =$ ____________________ . Here a may be either finite or infinite.

3.2 *Theorem: Cauchy's Mean Value Theorem*

Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \underline{\hspace{3cm}}.$$

✎ Ex. **7** .... (example7, p400)

Apply l'Hôpital's Rule to show that $\lim_{x\to 0^+}(1+x)^{1/x} = e$

sol:

✎ Ex. **8** .... (example8, p400)

Find $\lim\limits_{x\to\infty} x^{1/x}$.

sol:

實習課練習 (EXERCISE 7.5)

Use L'Hôpital Rule to find the limits.

5. $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$.

8. $\lim_{x\to -5} \dfrac{x^2-25}{x+5}$.

14. $\lim_{t\to 0} \dfrac{\sin 5t}{2t}$.

20. $\lim_{x\to 1} \dfrac{x-1}{\ln x - \sin \pi x}$.

27. $\lim_{\theta\to 0} \dfrac{3^{\sin\theta}-1}{\theta}$.

29. $\lim_{x\to 0} \dfrac{x2^x}{2^x-1}$.

34. $\lim_{x\to 0^+} \dfrac{\ln(e^x-1)}{\ln x}$.

41. $\lim_{x\to 1^+} (\dfrac{1}{x-1} - \dfrac{1}{\ln x})$.

46. $\lim_{x\to\infty} x^2e^{-x}$.

48. $\lim_{x\to 0} \dfrac{(e^x-1)^2}{x\sin x}$.

53. $\lim_{x\to\infty} (\ln x)^{1/x}$.

58. $\lim_{x\to 0} (e^x+x)^{1/x}$.

59. $\lim_{x\to 0^+} x^x$.

60. $\lim_{x\to 0^+} (1+\dfrac{1}{x})^x$.

62. $\lim_{x\to\infty} (\dfrac{x^2+1}{x+2})^{1/x}$.

73. $\lim_{x\to\infty} \dfrac{e^{x^2}}{xe^x}$.

THOMAS' CALCULUS (12/E)

7.6 Inverse Trigonometric Functions

開課班級: 資訊 1/電機 1/智財學程微積分
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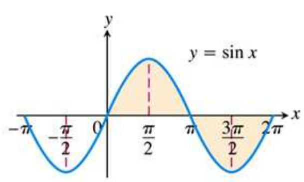
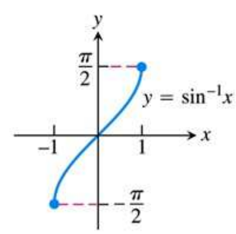
教學網站: http://www.hmwu.idv.tw

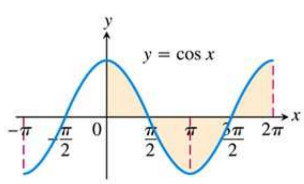
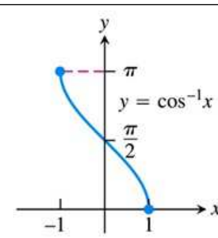
1 Defining the Inverses

1.1 Domain restrictions that make the trigonometric functions one-to-one.

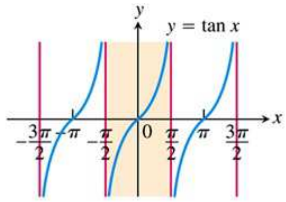
(a) $y = \sin^{-1} x$ is the number in ________________ for which ______________.

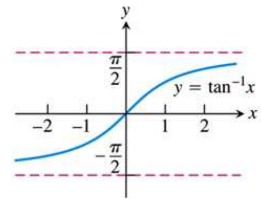
(b) $y = \cos^{-1} x$ is the number in __________ for which ______________.

$\sin x$	$\sin^{-1} x$ or arcsin$x$
D: ________________ R: ___________ 	D: ___________ R: ________________ 

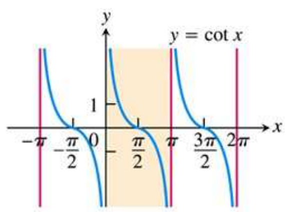
$\cos x$	$\cos^{-1} x$ or arccos$x$
D: __________ R: ____________ 	D: ___________ R: __________ 

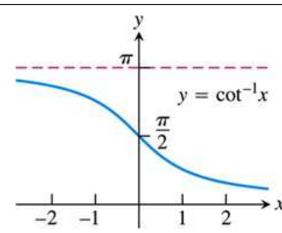
$\tan x$	$\tan^{-1} x$ or $\arctan x$
D: ________ R: ________	D: ________ R: ________



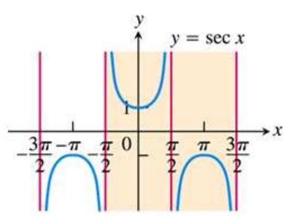


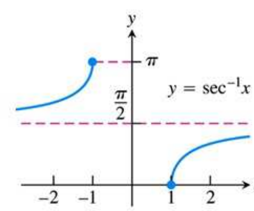
$\cot x$	$\cot^{-1} x$ or $\operatorname{arccot} x$
D: ________ R: ________	D: ________ R: ________



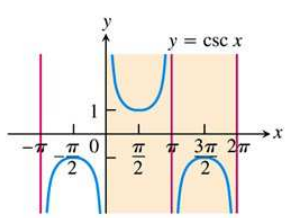


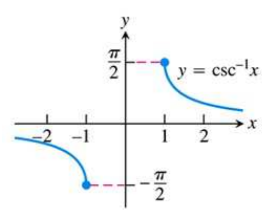
$\sec x$	$\sec^{-1} x$ or $\operatorname{arcsec} x$
D: ________ R: ________	D: ________ R: ________



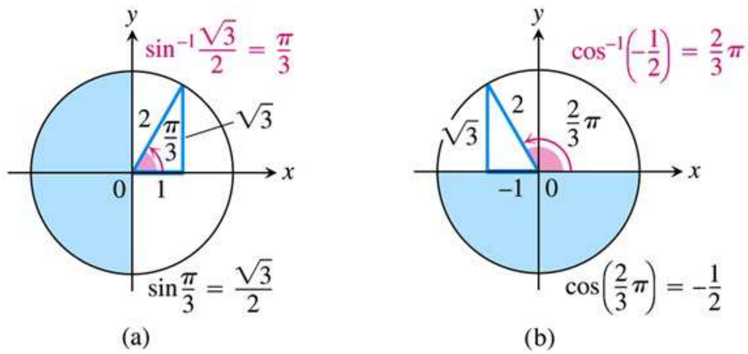


$\csc x$	$\csc^{-1} x$ or $\operatorname{arccsc} x$
D: ________ R: ________	D: ________ R: ________





1.2 Common values of $\sin^{-1} x$:



x	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$
$\sin^{-1} x$						
$\cos^{-1} x$						

TABLE 1.3 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

2 The Drivative of Inverse Trigonometric Functions

2.1 $(f(x)^{-1})' =$ ________________ .

2.2 Find the derivative of $y = \sin^{-1} x$.

2.3 Find the derivative of $y = \tan^{-1} x$.

2.4 Find the derivative of $y = \sec^{-1} x$.

2.5 $u = u(x)$

(a) $\dfrac{d}{dx}(\sin^{-1} u) = \underline{\qquad\qquad\qquad}$, $|u| < 1$.

(b) $\dfrac{d}{dx}(\tan^{-1} u) = \underline{\qquad\qquad\qquad}$.

(c) $\dfrac{d}{dx}(\sec^{-1} u) = \underline{\qquad\qquad\qquad\qquad\qquad}$.

2.6 *Inverse Function-Inverse Cofunction Identities*

(a) $\cos^{-1} x = \underline{\qquad\qquad\qquad} \Rightarrow \dfrac{d}{dx}(\cos^{-1} u) = \underline{\qquad\qquad\qquad\qquad}$, $|u| < 1$

(b) $\cot^{-1} x = \underline{\qquad\qquad\qquad} \Rightarrow \dfrac{d}{dx}(\cot^{-1} u) = \underline{\qquad\qquad\qquad}$

(c) $\csc^{-1} x = \underline{\qquad\qquad\qquad} \Rightarrow \dfrac{d}{dx}(\csc^{-1} u) = \underline{\qquad\qquad\qquad\qquad}$, $|u| > 1$

✎ Ex. **1** . . . . (`example4, p408`)

$\dfrac{d}{dx}(\sin^{-1} x^2) =$

sol:

✎ Ex. **2** . . . . (`example5, p410`)

$\dfrac{d}{dx}(\sec^{-1} 5x^4) =$

sol:

3 Integration Formulas

3.1 $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} =$ ________________, $u^2 < a^2$

3.2 $\displaystyle\int \frac{du}{a^2+u^2} =$ ________________

3.3 $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} =$ ________________, $|u| > a > 0$

✐ Ex. **3** .... (example6, p412)

(a) $\displaystyle\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} =$

(b) $\displaystyle\int \frac{dx}{\sqrt{3-4x^2}} =$

(c) $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-6}} =$

✎ Ex. **4** .... (example7(a), p412)

Evaluate $\int \frac{dx}{\sqrt{4x - x^2}}$

sol:

✎ Ex. **5** .... (example7(b), p412)

Evaluate $\int \frac{dx}{4x^2 + 4x + 2}$

sol:

實習課練習 (EXERCISE 7.6)

☐ Find the values.

9. $\sin(\cos^{-1}(\frac{\sqrt{2}}{2}))$

12. $\cot(\sin^{-1}(-\frac{\sqrt{3}}{2}))$

☐ Find the derivative of y with respect to the appropriate variable.

22. $y = \cos^{-1}(1/x)$ $y = \sin^{-1}(3/t^2)$

33. $y = \ln(\tan^{-1} x)$

41. $y = x \sin^{-1} x + \sqrt{1 - x^2}$

42. $y = \ln(x^2 + 4) - x \tan^{-1}(\frac{x}{2})$

☐ Evaluate the integrals.

44. $\int \frac{dx}{\sqrt{1 - 4x^2}}$

46. $\int \frac{dx}{9 + 3x^2}$

47. $\int \frac{dx}{x\sqrt{25x^2 - 2}}$

63. $\int_0^{\ln \sqrt{3}} \frac{e^x \, dx}{1 + e^{2x}}$

68. $\int \frac{dx}{\sqrt{2x - x^2}}$

72. $\int \frac{dy}{y^2 + 6y + 10}$

79. $\int \frac{dx}{(x + 1)\sqrt{x^2 + 2x}}$

83. $\int \frac{(\sin^{-1} x)^2 \, dx}{\sqrt{1 - x^2}}$

86. $\int \frac{dy}{(\sin^{-1} y)\sqrt{1 - y^2}}$

105-1 小考 (1) 題目

2016/10/03, Calculus Quiz (1), §2.1 ∼ §2.5 (交回題目卷及答案卷)

1. (a) What is the instantaneous speed at time t_0? (b) What is the Sandwich Theorem for ? (c) What is the precise definition of a limit? (d) What is the continuity of a function at a point?

2. Find the slope of the curve $y = 2 - x^3$ at the given point $P(1,1)$, and an equation of the tangent line at $P(1,1)$. (Hint: start with the slope of a secant to the graph $y = f(x)$)

3. If $\lim_{x\to 0} \frac{f(x)}{x^2} = 1$, find (a) $\lim_{x\to 0} f(x)$, (b) $\lim_{x\to 0} \frac{f(x)}{x}$. (Hint: 需有算式)

4. Use the precise definition of a limit to show that $\lim_{x\to c} \frac{1}{x} = \frac{1}{c}$, where c is a constant.

5. Find the limit: (a) $\lim_{t\to -3^+} \frac{\lfloor t \rfloor}{t}$, (b) $\lim_{t\to -3^-} \frac{\lfloor t \rfloor}{t}$. (c) $\lim_{t\to -4^+} (t - \lfloor t \rfloor)$, (d) $\lim_{t\to -4^-} (t - \lfloor t \rfloor)$
(Hint: $\lfloor x \rfloor$ is the greatest integer of x)(需有算式)

6. For what values of a and b is

$$g(x) = \begin{cases} ax + 2b, & x \le 0, \\ x^2 + 3a - b, & 0 < x \le 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuous at every x?

105-1 小考 (2) 題目

2016/10/24, Calculus Quiz (2), §2.5 ∼ §3.2 (交回題目卷及答案卷)(每題 20 分)

1. (a) What is the "Intermediate Value Theorem for Continuous Functions"? (b) Use the Intermediate Value Theorem to prove that the equation $x(x-1)^2 = 1$ has a solution.

2. Find the limit $\lim \left(\dfrac{1}{x^{1/3}} - \dfrac{1}{(x-1)^{4/3}} \right)$ as (a) $x \to 0^+$. (b) $x \to 0^-$. (c) $x \to 1^+$. (d) $x \to 1^-$.

3. Find the limit (a) $\lim\limits_{x\to-\infty}(2x + \sqrt{4x^2+3x-2})$. (b) $\lim\limits_{x\to\infty}(\sqrt{x^2+x} - \sqrt{x^2-x})$

4. (a) Find the horizontal and vertical asymptotes of the graph of $f(x) = \dfrac{1-x^2}{x^2+1}$. (b)Find the oblique asymptotes: $f(x) = \dfrac{x^2-4}{x-1}$.

5. (a) Using the definition to find the derivative of $f(x) = \sqrt{2x+1}$. (b) Find the tangent line to the above curve at $x = 1/2$.

105-1 小考 (3) 題目

2016/12/05, Calculus Quiz (3), §4.1 ∼ §4.6 (交回題目卷及答案卷)(每題 20 分)

1. (a) What is the Mean Value Theorem? (b) What is the definition of a critical point of a function f? (c) What is the definition of an inflection point of a function f?
2. Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.
3. Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area?
4. Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2. (ps. x_2 列式即可，不需算出答案)

105-1 小考 (4) 題目

2016/12/26, Calculus Quiz (4), §4.7 ∼ §5.6 (交回題目卷及答案卷)(每題 25 分)

1. (a) What is the definition of an "antiderivative"? (b) What is a Riemann for f on the interval $[a, b]$? (c) What are The Fundamental Theorem of Calculus, Part I and Part II?

2. (a) Compute $\int_0^b \pi x^2\, dx$ and (b) use the limit of Riemann sums to find the area under $y = \pi x^2$ over the interval $[0, b]$, $b > 0$.

3. (a) $\int_2^{-2} |x+1|\, dx$, (b) Find dy/dx where $y = x\int_2^{x^2} \sin(t^3)\, dt$.

4. (a) $\int \frac{1}{\sqrt{2x}(1+\sqrt{2x})^2}\, dx$, (b) Find the area of the region between the curve $y = 5 - x^2$ and the line $y = 1$.

國立臺北大學 105 學年度第 1 學期 期中 考試命題紙

考試科目： 微積分 開課班別： 通訊 1/電機 1/智財學程 命題教授: 吳漢銘

考試日期：11 月 7 日（一）10:00~11:30	※准帶項目打「O」，否則打「X」					1. 需加發計算紙或答案紙請在試題內封袋備註。 2. 為環保節能減碳，試題一律採雙面印刷，如有特殊印製需求，請註記：
本試題共 1 頁，印刷份數: 82 份	計算機	課本	筆記	電子辭典	紙本字典	
備註：注意事項要看!!	X	X	X	X	X	

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1. (a) What is the instantaneous rate of change of $f(x)$ at $x = c$. (b) What is the derivative of the function $f(x)$ with respect to x. (c) What does it mean that a function $f(x)$ is differentiable at $x = c$?

2. Let $f(x)$ and $g(x)$ be differentiable functions. Use the definition of the derivative of a function to prove The Quotient Rule.

3. Let $s(t) = \dfrac{t-1}{t+1}$. Use calculus (the definition of the derivative) to find the instantaneous rate of change of $s(t)$ at $t = -1/2$.

4. Find the derivative of f. (a) $f(x) = \sqrt[3]{(3x^2-4)^5 + 3x}$. (b) $f(x) = \dfrac{3x-2}{(2x-1)^2}$

5. Find $h'(0)$ if $h(x) = \left[\dfrac{g(x)-x}{3+g(x)}\right]^2$, where $g(0) = 3$ and $g'(0) = -2$.

6. (a) What is the definition of the continuity of a function f at the point c ? (b) What is the definition of the derivative of a function f at x? (c) What does "a function is differentiable at x" mean?

7. (a) Find $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} \dfrac{x}{3} - 2, & x \le 2 \\ -2x+5, & x > 2 \end{cases}$ (b) Find $\lim_{x\to -2} \dfrac{\lfloor 2x \rfloor + 2}{2x+2}$, where $\lfloor \cdot \rfloor$ is the Gauss' symbol.

8. Find dy/dx for the equation $xy^3 + (2y)^2 - 5y = \dfrac{(x^2-9)}{\sqrt{x^2+2}}$.

9. Find the asymptotes of the graph of $f(x) = \dfrac{1-x^2}{8-(\sqrt{2}x)^2}$.

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國立臺北大學 105 學年度第 1 學期 期末 考試命題紙

考試科目： 微積分　　開課班別： 通訊 1/電機 1/智財學程　　命題教授: 吳漢銘

考試日期：1 月 9 日（一）10:00∼11:30	※准帶項目打「O」，否則打「X」					1. 需加發計算紙或答案紙請在試題內封袋備註。 2. 為環保節能減碳，試題一律採雙面印刷，如有特殊印製需求，請註記：
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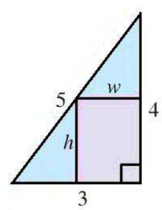
1. (10 分) (a) What is a Riemann sum for a function f? (b) What are "The Fundamental Theorem of Calculus" Part 1 and Part 2?

2. (15 分) Sketch the graph of $f(x) = \dfrac{x^2-3}{x-2}$. (Find where the curve is increasing and where it is decreasing. Find the local extreme points. Find the points of inflection , if any occur, and determine the concavity of the curve. Identify any asymptotes.)

3. (10 分) Determine the dimensions of the rectangle of largest area that can be inscribed in the right triangle shown in the following figure.



4. (15 分) (a) Compute $\int_a^b x^2\,dx$. (b) Use the limit of Riemann sums to find the area under $y = x^2$ over the interval $[a, b]$, $b > a$.

5. (10 分) Find dy/dx, if $y = \int_2^{7x^2} \sqrt{2+\cos^3 t}\,dt$.

6. (10 分) Find the areas of the regions enclosed by the lines and curves. $y = |x^2-4|$, and $y = (x^2/2)+4$.

7. (10 分) $\int x^5\sqrt{2x^3+1}\,dx$.

8. (20 分) By integration, find the volume of the solid generated by revolving the triangular region with vertices $(0, 0), (b, 0), (0, h)$ about (a) the x-axis. (b) the y-axis.

9. 加分題 (20 分): (a) 微積分助教姓名? (b) 助教研究室在哪一間? (c) 微積分課本書名為何? (d) 微積分課本作者及版本為何?

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106-1 小考 (1) 題目

2017/10/02, Calculus Quiz (1), §2.1 ∼ §2.5 (可用鉛筆、需計算過程、交回題目卷及答案卷)(每題 20 分)

1. (a) What is the instantaneous speed at time t_0? (b) What is the Sandwich Theorem? (c) What is the continuity of a function at a point? (d) What is the Intermediate Value Theorem for Continuous Functions?

2. Can $f(x) = x(x^2-1)/|x^2-1|$ be extended to be continuous at $x = 1$ or -1? Give reasons for your answers.

3. Let $f(x) = x^3 - 2x + 2$. Use the Intermediate Value Theorem to show that f has a zero between -2 and 0.

4. (a) Find $\lim\limits_{x\to 0^+} g(x)$ if $\lim\limits_{x\to 0^+} (4g(x))^{1/3} = 2$. (b) Find $\lim\limits_{x\to\sqrt{5}} g(x)$ if $\lim\limits_{x\to\sqrt{5}} \dfrac{1}{x+g(x)} = 2$.

5. Use the precise definition of a limit to show that $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = 2$.

106-1 小考 (2) 題目

2017/10/25, Calculus Quiz (2), §2.6 ~ §3.5 (需計算過程、交回題目/答案卷)(每題 20 分)

1. Find the all asymptotes of the graph of $f(x)$. (a) $f(x) = \dfrac{1-x^2}{x^2+1}$. (b) $f(x) = \dfrac{x^2-1}{2x+4}$.

2. (a) What is the definition of the derivative of a function f at a point x_0?
   (b) State and prove that the Differentiability implies Continuity.

3. Using the definition, calculate the derivatives of the functions.
   (a) $f(x) = \sqrt{2x+1}$. (b) $g(x) = \dfrac{x}{x-1}$.

4. Find the derivative of $y = \dfrac{(\sin x + 1)(\cos x + 2)}{(\tan x - 1)(\sec x - 2)}$.

5. Evaluate each limit: $\lim\limits_{x \to -1} \dfrac{x^{2/9} - 1}{x+1}$.

106-1 小考 (3) 題目

2017/12/18, Calculus Quiz (3), §4.1 ∼ §4.6 (交回題目卷及答案卷)(每題 25 分)

1. (a) What is the Mean Value Theorem? (b) What is the definition of a critical point of a function f? (c) What is the definition of an inflection point of a function f?

2. Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

3. Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area?

4. Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2. (ps. x_2 列式即可，不需算出答案)

106-1 小考 (4) 題目

2017/12/25, Calculus Quiz (4), §4.7 ∼ §5.6 (交回題目卷及答案卷)(每題 25 分)

1. (a) What is the definition of an "antiderivative"? (b) What is a Riemann for f on the interval $[a, b]$? (c) What are The Fundamental Theorem of Calculus, Part I and Part II?

2. (a) Compute $\displaystyle\int_0^b \pi x^2\, dx$ and (b) use the limit of Riemann sums to find the area under $y = \pi x^2$ over the interval $[0, b]$, $b > 0$.

3. (a) $\displaystyle\int_2^{-2} |x+1|\, dx$, (b) Find dy/dx where $y = x\displaystyle\int_2^{x^2} \sin(t^3)\, dt$.

4. (a) $\displaystyle\int \frac{1}{\sqrt{2x}(1+\sqrt{2x})^2}\, dx$, (b) Find the area of the region between the curve $y = 5 - x^2$ and the line $y = 1$.

國立臺北大學 106 學年度第 1 學期 期中 考試命題紙						
考試科目： 微積分 開課班別： 通訊 1/電機 1/智財學程 命題教授: 吳漢銘						
考試日期：11 月 06 日 (一) 10:00~11:30	※准帶項目打「O」，否則打「×」					1. 需加發計算紙或答案紙請在試題內封袋備註。 2. 為環保節能減碳，試題一律採雙面印刷，如有特殊印製需求，請註記：
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備註：注意事項要看!!	×	×	×	×	×	

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1. (10 分) (a) What is the continuity of a function at a point? (b) State and prove that the Differentiability implies Continuity.

2. (10 分) Can $f(x) = \dfrac{x(x^2-1)}{|x^2-1|}$ be extended to be continuous at $x = 1$ or -1? Give reasons for your answers.

3. (20 分) Find the limits: (a) $\lim_{x\to 1} \dfrac{x^{50}-1}{x-1}$. (b) $\lim_{x\to\infty} \dfrac{x+\sin x + 2\sqrt{x}}{x+\sin x}$.

4. (10 分) Find $\lim_{x\to -2} \dfrac{\lfloor 2x \rfloor + 2}{2x+2}$, where $\lfloor\cdot\rfloor$ is the Gauss' symbol.
   ($y = [x]$: the function whose value at any number x is the greatest integer less than or equal to x.)

5. (10 分) At what points is the function $y = |x-1| + \sin x$ continuous?

6. (10 分) Use the Chain Rule to find the value of $(f \circ g)'$ at the given value of x:
$$f(u) = (\frac{2u-1}{2u+1})^2,\ u = g(x) = \frac{1}{x^2} - 1,\ x = -1.$$

7. (20 分) Use implicit differentiation to find
   (a) dy/dx if $x\cos(2x+3y) = y\sin 5x$.
   (b) d^2y/dx^2 if $y^2 = 1 - \dfrac{2}{x}$.

8. (10 分) (a) Find the linearization of $f(x) = (1+2x)^k$ at $x = 0$. (b) Use the approximation in (a) to estimate $(1.0002)^{50}$.

9. (20 分) Let $f(x)$ and $g(x)$ be differentiable functions. Use the definition of the derivative of a function to prove The Derivative Quotient Rule.

國立臺北大學 106 學年度第 1 學期 期末 考試命題紙

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備註：注意事項要看!!	×	×	×	×	×	

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1. (10 分) Use the limit of Riemann sums to find the area of the region between the curve $y = 3x^2$ and the x-axis on the interval $[0, b]$.

2. (20 分) Sketch the graph of $f(x) = \dfrac{x^2 + 4}{2x}$. (提示: 有斜漸近線) (identify relative extrema, infliection points, concavity, and asymptotes if any.)

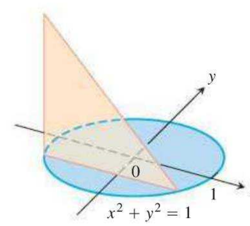
3. (10 分) Find dy/dx if $y = \left(\int_0^{5x} (t^3 + 1)^{10}\ dt \right)^3$.

4. (10 分) Evaluate the integral: $\int_1^4 \dfrac{dy}{2\sqrt{y}(1 + \sqrt{y})^2}$.

5. (10 分) Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

6. (10 分)

   The base of the solid is the disk $x^2 + y^2 \leq 1$. The cross-sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles (等腰的) right triangles with one leg in the disk. Find the volumes of the solids.



7. (10 分) Find the volumes of the solids generated by revolving the regions bounded by the lines $x = 0$, $y = 1$ and curve $x = \sqrt{2y}/(y^2 + 1)$ about the y-axis.

8. (20 分) Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about (a) the line $y = 2$. (b) the line $x = 4$.

9. (20 分) (a) Derive the formula of The Derivative Rule for Inverse. (b) Let $f(x) = x^2 - 4x - 5$, $x > 2$. Use the above formula to find the value of df^{-1}/dx at the point $x = 0 = f(5)$.

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