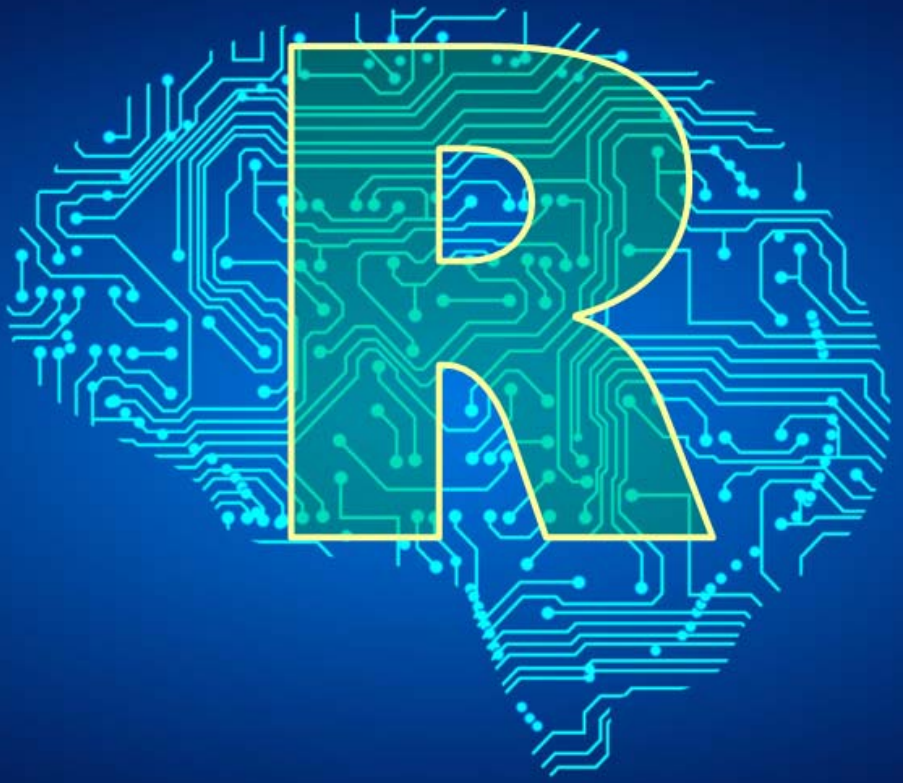
StatR-05

台灣人工智慧學校



無母數統計

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無母數統計 - 大綱

- **主題1**
  - **Non-parametric Models**
  - **Non-parametric Tests**
    - Sign Test，Wilcoxon Signed-Rank Test (paired), Mann-Whitney Test, Kruskal-Wallis Test
  - **事後比較檢定 (Post Hoc Tests):** Tukey's HSD Test
- 主題2
  - 常態分佈檢定 (Test for Normality)
  - 卡方檢定 (Chi-Square Test)

無母數統計
Non-parametric Statistics

- Nonparametric statistics is based on either
  - being distribution-free or having a specified distribution but with the distribution's parameters unspecified.
  - includes both descriptive statistics and statistical inference.
- **Non-parametric inferential statistical methods**: Sign test, Wilcoxon signed-rank test, Mann–Whitney U test, Kolmogorov–Smirnov test, Kruskal–Wallis one-way ANOVA,...
- **Non-parametric models**: kernel density estimation, non-parametric regression, ...

kernel regression

$$\hat{f}(x) = \frac{\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right) y_i}{\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right)}$$

k-nearest-neighbors regression.

$$\hat{f}(x) = \frac{1}{k} \sum_{i \in \mathcal{N}_k(x)} y_i$$

nonparametric regression

$$y_i = f_0(x_i) + \epsilon_i, \quad i = 1, \ldots n,$$

$\epsilon_1, \ldots \epsilon$ are still i.i.d. random errors with $\mathbb{E}(\epsilon_i) = 0$

https://en.wikipedia.org/wiki/Nonparametric_statistics

平均數檢定 in R

Hypothesis Testing	One Sample	Two Samples		> two Groups
	-	Paired data	Unpaired data	Complex data
Parametric (variance equal)	t-test t.test(x, mu = 0)	t-test t.test(x-y, var.equal = TRUE) t.test(x, y, paired = TRUE, var.equal = TRUE)	t-test t.test(x, y, var.equal = TRUE)	One-Way Analysis of Variance (ANOVA) aov(x~g, data) oneway.test(x~g, data, var.equal = TRUE)
Parametric (variance not equal)		Welch t-test t.test(x-y) t.test(x, y, paired = TRUE)	Welch t-test t.test(x, y)	Welch ANOVA oneway.test(x~g, data)
Non-Parametric (無母數檢定)	Wilcoxon Signed-Rank Test wilcox.test(x, mu = 0)	Wilcoxon Signed-Rank Test wilcox.test(x-y) wilcox.test(x, y, paired = TRUE)	Wilcoxon Rank-Sum Test (Mann-Whitney U Test) wilcox.test(x, y)	Kruskal-Wallis Test kruskal.test(x, g)

pairwise.t.test {stats}: Calculate pairwise comparisons between group levels with corrections for multiple testing

TukeyHSD {stats}: Compute Tukey Honest Significant Differences

Sign Test

- Given *n* pairs of data, the sign test tests the hypothesis that the **median** of the differences in the pairs is zero.
- The test statistic is the number of positive differences.
- If the null hypothesis is true, then the numbers of positive and negative differences should be approximately the same.

Pair	Before	After	Sign
1	89	73	+
2	83	77	+
3	80	58	+
4	72	77	−
5	77	70	+
6	74	62	+
7	69	67	+
8	65	68	−
9	60	44	+
10	55	50	+
11	54	46	+
12	50	38	+
13	42	47	−
14	48	40	+
15	44	43	+
16	38	29	+
17	36	25	+

The Sign Test:

when $n_1 = n_2 \leq 50$

$H_0 : P = Q = \frac{1}{2}$

$H_1 : P \neq Q \neq \frac{1}{2}$

$T = \#"+"$

At $\alpha = 0.01$, two-tailed test,

reject H_0 if $T \geq 14$ when $N = 17$.

(Binomial Probability)

Wilcoxon Signed-Rank Test (paired)

- Null hypothesis: the **population median** from which both samples were drawn is the same.
- The sum of the ranks for the "positive" values is calculated and compared against a precomputed table to a p-value.
- If the null hypothesis is true, the sum of the ranks of the positive differences should be about the same as the sum of the ranks of the negative differences.

Pair	Before	After	Diff.	Rank
1	89	73	16	15.5
2	83	77	6	7
3	80	58	22	17
4	72	77	−5	5
5	77	70	7	8
6	74	62	12	13.5
7	69	67	2	2
8	65	68	−3	3
9	60	44	16	15.5
10	55	50	5	5
11	54	46	8	9.5
12	50	38	12	13.5
13	42	47	−5	5
14	48	40	8	9.5
15	44	43	1	1
16	38	29	9	11
17	36	25	11	12

The Wilcoxon signed-rank Test:

$H_0 : \mu_1 = \mu_2$
$H_1 : \mu_1 \neq \mu_2$

$T = \min\{\sum_+ \text{Rank}, \sum_- \text{Rank}\}$

At $\alpha = 0.01$, two-tailed test,
reject H_0 if $T \neq 23$ when $N = 17$.
(Table)

(The zero difference is ignored when assigning ranks. $N_{new} = N_{old} - \#\{ties\}$)

$T = \min\{\sum_+ \text{Rank} = 140, \sum_- \text{Rank} = 13\}$
$= 13$

The obtained T=13 is less than the critical value 23, so we reject H_0.

Mann-Whitney Test

(Wilcoxon Rank-Sum Test, unpaired)

- The data from the two groups are combined and given ranks. (1 for the largest, 2 for the second largest,...)

- The ranks for the larger group are summed and that number is compared against a precomputed table to a p-value.

Group		Rank	
G_1	G_2	G_1	G_2
26	16	3	11
22	10	4	17
19	8	7.5	19
21	13	5.5	13.5
14	19	12	7.5
18	11	9	15.5
29	7	2	20
17	13	10	13.5
11	9	15.5	18
34	21	1	5.5
$n_1 = 10$	$n_2 = 10$	$R_1 = 69.5$	$R_2 = 140.5$

The Mann-Whitney U Test:

$H_0 : F_1 = F_2$

$H_1 : F_1 \neq F_2$

$U = n_1 n_2 + \frac{n_1(n_1+1)}{2} - R_1$

or

$U' = n_1 n_2 + \frac{n_2(n_2+1)}{2} - R_2$

$R_i = \sum_i \text{Rank}$

At $\alpha = 0.05$, two-tailed test for $n_1 = 10, n_2 = 10$, reject H_0 if $U \leq 23$ or $U' \geq 77$ (Table)

U : the number of times that a score from Group 1 is lower in rank than a score from Group 2.

$U = 85.5, \quad U' = 14.5$

The obtained $U = 85.5$ is less than the critical value 77, so we reject H_0.

Kruskal-Wallis Test

- The Kruskal Wallis test can be applied in the one factor ANOVA case. It is a non-parametric test for the situation where the ANOVA normality assumptions may not apply.
- Each of the n_i should be at least 5 for the approximation to be valid.

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$$

$$H_1 : \mu_i \neq \mu_j \quad \text{for at least one set of } i \text{ and } j$$

$$W = \frac{12}{N(N+1)} \sum_{j=1}^{k} \frac{R_j^2}{n_j} - 3(N+1)$$

$$W \sim \chi^2_{k-1} \text{ under } H_0$$

Reject H_0 if $W > CHIPPF(\alpha, k-1)$, the chi-square percent point function

Groups

1	2	...	j	...	k
X_{11}	X_{12}	$\cdots$	X_{1j}	$\cdots$	X_{1k}
X_{21}	X_{22}	$\cdots$	X_{2j}	$\cdots$	X_{2k}
			$\cdots$		
X_{i1}	X_{i2}	$\cdots$	X_{ij}	$\cdots$	X_{ik}
$\vdots$	$X_{n_2 2}$	$\cdots$	$\vdots$	$\cdots$	$X_{n_k k}$
$X_{n_1 1}$			$X_{n_i j}$		

Rank Data

1	2	...	j	...	k
R_{11}	R_{12}	$\cdots$	R_{1j}	$\cdots$	R_{1k}
R_{21}	R_{22}	$\cdots$	R_{2j}	$\cdots$	R_{2k}
			$\cdots$		
R_{i1}	R_{i2}	$\cdots$	R_{ij}	$\cdots$	R_{ik}
$\vdots$	$R_{n_2 2}$	$\cdots$	$\vdots$	$\cdots$	$R_{n_k k}$
$R_{n_1 1}$			$R_{n_i j}$		

$$F(x) = P(X \leq x) = P(X \leq G(\alpha)) = \alpha$$

$$x = G(\alpha) = G(F(x))$$

The percent point function (ppf) is the inverse of the cumulative distribution function.

Parametric vs. Non-Parametric Test

Parametric Tests

- Assume that the data follows a certain distribution (normal distribution).
- Assuming equal **variances** and unequal variances.
- **More powerful.**
- Widely Implemented.
- Not appropriate for data with outliers.

Non-Parametric Tests

- When certain assumptions about the underlying population are questionable (e.g. normality).
- Does not assume normal distribution
- No variance assumption
- Less powerful.
- Widely Implemented.
- Decrease effects of outliers (Robust)
- Not recommended if there is less than 5 replicates per group.

事後檢定 (Post Hoc Tests)
Tukey's Honestly Significant Difference (HSD) Test

- **Null hypothesis**: all means being compared are from the same population (i.e. $\mu_1 = \mu_2 = \mu_3 = \ldots = \mu_k$)

$$q_s = \frac{Y_A - Y_B}{SE},$$

Y_A is the larger of the two means being compared,
Y_B is the smaller of the two means being compared, and
SE is the standard error of the sum of the means.

- This q_s value can then be compared to a ***q value*** from the studentized range distribution.
- If the q_s value is larger than the critical value q_α obtained from the distribution, the two means are said to be significantly different at level α , $0 \le \alpha \le 1$.

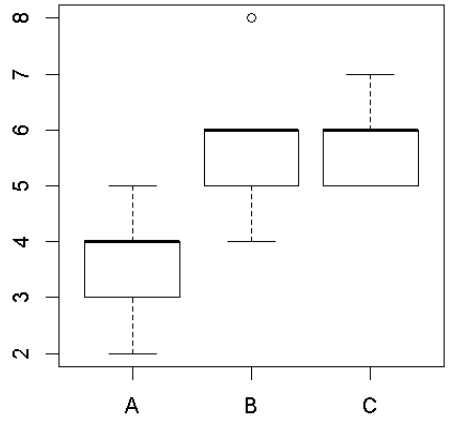
- **Assumptions for the test**
  - Observations are **independent** within and among groups.
  - The groups for each mean in the test are **normally distributed**.
  - **equal within-group variance** across the groups.
  - equal **sample sizes**.

範例: ANOVA + Post Hoc Test

- A drug company tested three formulations of a pain relief medicine for migraine headache sufferers. For the experiment 27 volunteers were selected and 9 were randomly assigned to one of three drug formulations. The subjects were instructed to take the drug during their next migraine headache episode and to report their pain on a scale of 1 to 10 (10 being most pain).

Drug A:	4	5	4	3	2	4	3	4	4
Drug B:	6	8	4	5	4	6	5	8	6
Drug C:	6	7	6	6	7	5	6	5	5

```
> pain <- c(4, 5, 4, 3, 2, 4, 3, 4, 4, 6, 8, 4, 5,
+  4, 6, 5, 8, 6, 6, 7, 6, 6, 7, 5, 6, 5, 5)
> drug <- c(rep("A", 9), rep("B", 9), rep("C", 9))
> migraine <- data.frame(pain, drug)
> plot(pain ~ drug, data=migraine)
> migraine.aov <- aov(pain ~ drug, data=migraine)
> summary(migraine.aov)
            Df Sum Sq Mean Sq F value   Pr(>F)
drug         2  28.22  14.111   11.91 0.000256 ***
Residuals   24  28.44   1.185
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
> # reject the null hypothesis of equal means for all three drug grou
```




```
> kruskal.test(pain ~ drug, data=migraine)
        Kruskal-Wallis rank sum test
data:  pain by drug
Kruskal-Wallis chi-squared = 14.395, df = 2, p-value = 0.0007483
```

Pairwise Comparisons

```
> pairwise.t.test(pain, drug, p.adjust="bonferroni")

        Pairwise comparisons using t tests with pooled SD

data:  pain and drug

  A       B
B 0.00119 -
C 0.00068 1.00000

P value adjustment method: bonferroni
>
> TukeyHSD(migraine.aov)
  Tukey multiple comparisons of means
    95% family-wise confidence level

Fit: aov(formula = pain ~ drug, data = migraine)

$drug
         diff        lwr      upr     p adj
B-A 2.1111111  0.8295028 3.392719 0.0011107
C-A 2.2222222  0.9406139 3.503831 0.0006453
C-B 0.1111111 -1.1704972 1.392719 0.9745173
>
> # conclude that the mean pain is significantly different for drug A
```

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  - Non-parametric Models
  - Non-parametric Tests
    - Sign Test，Wilcoxon Signed-Rank Test (paired), Mann-Whitney Test, Kruskal-Wallis Test
  - 事後比較檢定 (Post Hoc Tests): Tukey's HSD Test

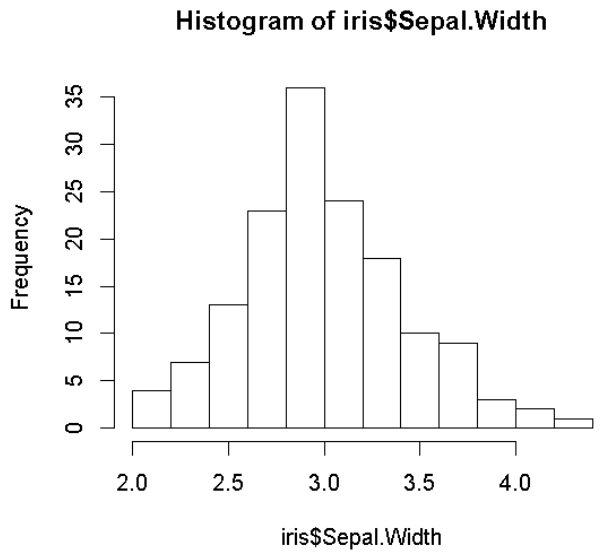
- 主題2
  - 常態分佈檢定 (Test for Normality)
  - 卡方檢定 (Chi-Square Test)

Formal Tests for Normality

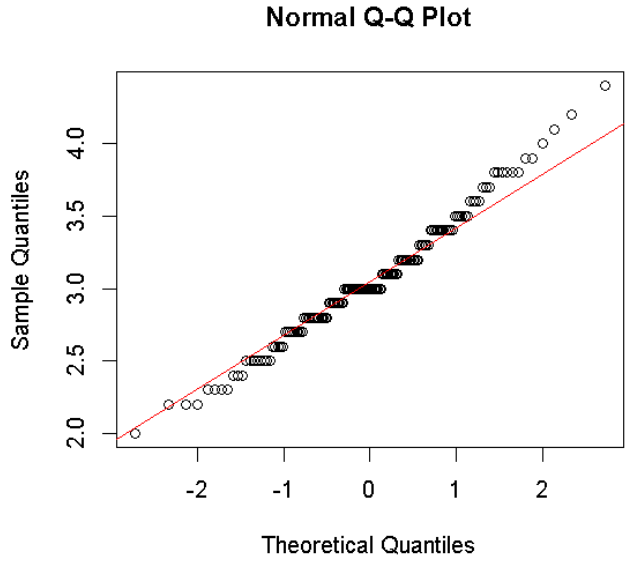
H_0: The sample data are **not** significantly **different** than a normal population.

H_a: The sample data are significantly different than a normal population

```
hist(iris$Sepal.Width)
```




```
qqnorm(iris$Sepal.Width)
qqline(iris$Sepal.Width, col="red")
```



nortest Packages: Tests for Normality

- **nortest** Packages: five omnibus tests for testing the composite hypothesis of normality: **ad.test, cvm.test, lillie.test, pearson.test, sf.test**
- **Other tests:**
  - Kolmogorov-Smirnov (K-S) test (Chakravarti et al., 1967).
  - The Shapiro-Wilk normality test (Shapiro and Wilk, 1965).

```
> x <- iris$Sepal.Width
> ks.test(x, 'pnorm', mean(x), sd(x))

        One-sample Kolmogorov-Smirnov test

data:  x
D = 0.10566, p-value = 0.07023
alternative hypothesis: two-sided

Warning message:
In ks.test(x, "pnorm", mean(x), sd(x)) :
  ties should not be present for the Kolmogorov-Smirnov test
```

```
> library(nortest)
> ad.test(iris$Sepal.Width)

        Anderson-Darling normality test

data:  iris$Sepal.Width
A = 0.90796, p-value = 0.02023
```

```
> shapiro.test(iris$Sepal.Width)

        Shapiro-Wilk normality test

data:  iris$Sepal.Width
W = 0.98492, p-value = 0.1012
```

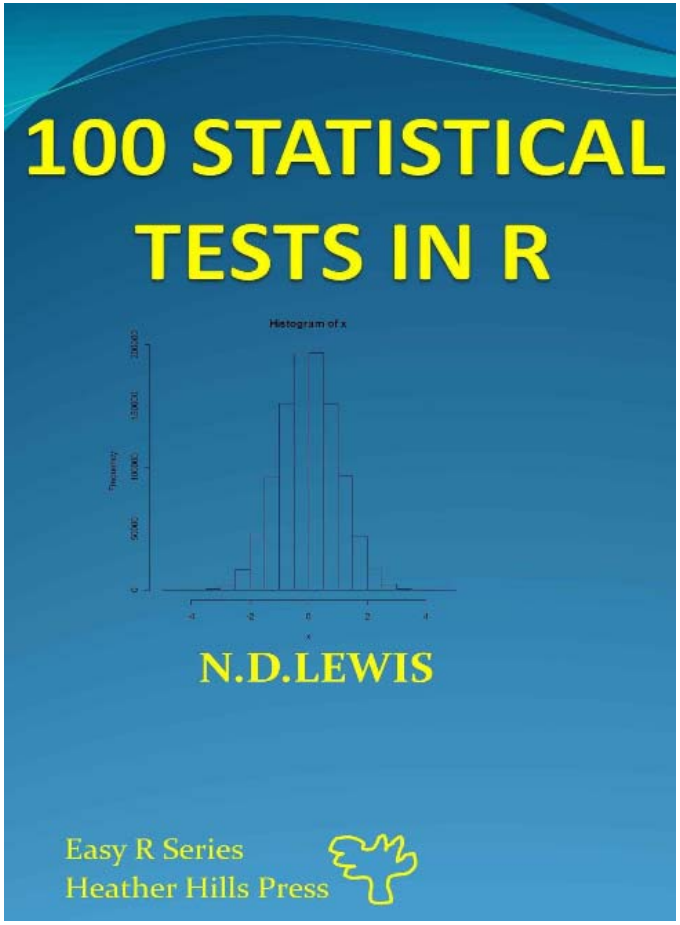
Which Normality Test Should I Use?

- **Kolmogorov-Smirnov test**:
  - It is more sensitive near the center of the density than at the tails than other tests;
  - For data sets n > 50.
- **The Anderson-Darling test**:
  - A-D test is a modification of the K-S test and gives more weight to the tails of the density than does the K-S test.
  - It is generally preferable to the K-S test.
- **Shapiro-Wilks test**:
  - Doesn't work well if several values in the data set are the same.
  - Works best for data sets with n < 50, but can be used with larger data sets.
- **W/S test (range(x)/sd(x))**:
  - simple, but effective.
- **Jarque-Bera test** (`jarque.test {moments}`):
  - tests for skewness and kurtosis, very effective.
- **D'Agostino test** (`agostino.test{moments}`) :
  - powerful omnibus (skewness, kurtosis, centrality) test.

Which Normality Test Should I Use?

- Asghar Ghasemi and Saleh Zahediasl, Normality Tests for Statistical Analysis: **A Guide for Non-Statisticians**,
  Int J Endocrinol Metab. 2012 Spring; 10(2): 486–489.
  - assessing the normality assumption should be taken into account for using parametric statistical tests.
  - The KS test, should no longer be used owing to its low power.
  - It is preferable that normality be assessed both visually and through normality tests, of which the **Shapiro-Wilk test** is highly recommended.
- **NOTE**:
  - If the data are not normal, use non-parametric tests.
  - If the data are normal, use parametric tests.
  - If you have groups of data, you MUST test each group for normality.
  - It's common seen that a model is built from the training data and is then applied to the testing data. Did these two data sets follow the same distribution?

卡方檢定: chisq.test



N.D Lewis, 100 Statistical Tests in R, Publisher: CreateSpace Independent Publishing Platform (April 15, 2013)

卡方檢定: chisq.test

- **適合度檢定**(test of goodness of fit): 檢定資料是否符合某個比例關係或某個機率分佈。
- **齊一性檢定**(test of homogeneity): 檢定幾個不同類別中的比例關係是否一致。
- **獨立性檢定**(test of independence): 檢定兩個分類變數之間是否互相獨立。

chisq.test {stats}: Pearson's Chi-squared Test for Count Data

Description:

chisq.test performs chi-squared contingency table tests and goodness-of-fit tests.

Usage:

```
chisq.test(x, y = NULL, correct = TRUE, p =
rep(1/length(x), length(x)), rescale.p = FALSE,
simulate.p.value = FALSE, B = 2000)
```

Chi-Square Test for Independence

H_0: In the population, the two categorical variables are **independent.**

For testing independence in $I \times J$ contingency tables

$$H_0: \pi_{ij} = \pi_{i+}\pi_{+j} \quad \text{for all } i \text{ and } j$$

$\mu_{ij} = n\pi_{ij} = n\pi_{i+}\pi_{+j}$ as the expected frequency.

estimated expected frequencies.

$$\hat{\mu}_{ij} = np_{i+}p_{+j} = n\left(\frac{n_{i+}}{n}\right)\left(\frac{n_{+j}}{n}\right) = \frac{n_{i+}n_{+j}}{n}$$

The *Pearson chi-squared statistic* for testing H_0 is

$$X^2 = \sum \frac{(n_{ij} - \mu_{ij})^2}{\mu_{ij}}$$

The X^2 statistic has approximately a chi-squared distribution, for large n. **(WHY?)**

Table 2.5. Cross Classification of Party Identification by Gender

	Party Identification			
Gender	Democrat	Independent	Republican	Total
Females	762 (703.7)	327 (319.6)	468 (533.7)	1557
Males	484 (542.3)	239 (246.4)	477 (411.3)	1200
Total	1246	566	945	2757

Note: Estimated expected frequencies for hypothesis of independence in parentheses. Data from 2000 General Social Survey.

```
> M <- as.table(rbind(c(762, 327, 468),
                      c(484, 239, 477)))
> dimnames(M) <- list(gender = c("F", "M"),
+                     party = c("Democrat",
                               "Independent",
                               "Republican"))
> M
      party
gender Democrat Independent Republican
     F      762         327        468
     M      484         239        477
> (res <- chisq.test(M))
        Pearson's Chi-squared test

data:  M
X-squared = 30.07, df = 2, p-value = 2.954e-07
```